Dispersion and limit theorems for random walks associated with hypergeometric functions of type BC

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Abstract

The spherical functions of the noncompact Grassmann manifolds $G_{p,q}(\mathbb{F}) = G/K$ over the (skew-)fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with rank $q \geq 1$ and dimension parameter $p > q$ can be described as Heckman-Opdam hypergeometric functions of type BC, where the double coset space $G//K$ is identified with the Weyl chamber $C_q^B \subset \mathbb{R}^q$ of type B. The corresponding product formulas and Harish-Chandra integral representations were recently written down by M. Rösler and the author in an explicit way such that both formulas can be extended analytically to all real parameters $p \in [2q-1, \infty[$, and that associated commutative convolution structures $*_p$ on C_q^B exist. In this paper we study the associated moment functions and the dispersion of probability measures on C_q^B with the aid of this generalized integral representation. This leads to strong laws of large numbers and central limit theorems for associated time-homogeneous random walks on $(C_q^B, *_p)$ where the moment functions and the dispersion appear in order to determine drift vectors and covariance matrices of these limit laws explicitely. For integers p, all results have interpretations for G-invariant random walks on the Grassmannians G/K.

Besides the BC-cases we also study the spaces $GL(q, \mathbb{F})/U(q, \mathbb{F})$, which are related to Weyl chambers of type A, and for which corresponding results hold. For the rank-one-case $q = 1$, the results of this paper are well-known in the context of Jacobi-type hypergroups on $[0, \infty[$.

Key words: Hypergeometric functions associated with root systems, Heckman-Opdam theory, non-compact Grassmann manifolds, spherical functions, random walks on symmetric spaces, random walks on hypergroups, dispersion, moment functions, central limit theorems, strong laws of large numbers.
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1 Introduction

The Heckman-Opdam theory of hypergeometric functions associated with root systems generalizes the classical theory of spherical functions on Riemannian symmetric spaces; see [H], [HS] and [O1] for the general theory, and [NPP], [R2], [RKV], [RV1], [Sch] for some recent developments. In this paper we study these functions for the root systems of types A and BC in the noncompact case. In the case A_{q-1} with $q \geq 2$, this theory is connected with the groups $G := GL(q, \mathbb{F})$ with maximal compact subgroups $K := U(q, \mathbb{F})$ over one of the (skew-)fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with dimension

$$d := \dim_{\mathbb{R}} \mathbb{F} \in \{1, 2, 4\} \qquad \text{for} \qquad \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}.$$

Moreover, in the case BC_q with $q \geq 1$, these functions are related with the non-compact Grassmann manifolds $\mathcal{G}_{p,q}(\mathbb{F}) := G/K$ with $p > q$, where depending on $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the group G is one of the indefinite orthogonal, unitary or symplectic groups $SO_0(q, p)$, $SU(q, p)$ or $Sp(q, p)$, and K is the maximal compact subgroup $K = SO(q) \times SO(p)$, $S(U(q) \times U(p))$ or $Sp(q) \times Sp(p)$, respectively.

In all these group cases, we regard the K-spherical functions on G (i.e., the nontrivial, K-biinvariant, multiplicative continuous functions on G) as multiplicative continuous functions on the double coset space $G//K$ where $G//K$ is equipped with the corresponding double coset convolution. By the KAK-decomposition of G in the both cases above, the double coset space $G//K$ may be identified with the Weyl chambers

$$C_q^A := \{t = (t_1, \cdots, t_q) \in \mathbb{R}^q : \ t_1 \geq t_2 \geq \cdots \geq t_q\}$$

of type A and

$$C_q^B := \{t = (t_1, \cdots, t_q) \in \mathbb{R}^q : \ t_1 \geq t_2 \geq \cdots \geq t_q \geq 0\}$$

of type B respectively. In both cases, this identification occurs via a exponential mapping $t \mapsto a_t \in G$ from the Weyl chamber to a system of representatives a_t of the double cosets in G. We now follow the notation in [RV1] and put

$$a_t = e^{\underline{t}} \tag{1.1}$$

for $t \in C_q^A$ in the A-case, and

$$a_t = \exp(H_t) = \begin{pmatrix} \cosh \underline{t} & \sinh \underline{t} & 0 \\ \sinh \underline{t} & \cosh \underline{t} & 0 \\ 0 & 0 & I_{p-q} \end{pmatrix} \tag{1.2}$$

for $t \in C_q^B$ in the BC-case respectively where we use the diagonal matrices

$$e^{\underline{t}} := \mathrm{diag}(e^{t_1}, \ldots, e^{t_q}),\ \cosh \underline{t} = \mathrm{diag}(\cosh t_1, \ldots, \cosh t_q),\ \sinh \underline{t} = \mathrm{diag}(\cosh t_1, \ldots, \cosh t_q).$$

We use this identification of $G//K$ and the corresponding Weyl chambers C_q^A or C_q^B from now on.

To identify the spherical functions, we fix the rank q, follow the notation in the first part of [HS], and denote the Heckman-Opdam hypergeometric functions associated with the root systems

$$2 \cdot A_{q-1} = \{\pm 2(e_i - e_j) : 1 \leq i < j \leq q\} \subset \mathbb{R}^q$$

and

$$2 \cdot BC_q = \{\pm 2e_i, \pm 4e_i, \pm 2e_i \pm 2e_j : 1 \leq i < j \leq q\} \subset \mathbb{R}^q$$

by $F_A(\lambda, k; t)$ and $F_{BC}(\lambda, k; t)$ respectively with spectral variable $\lambda \in \mathbb{C}^q$ and multiplicity parameter k. The factor 2 in the root systems originates from the known connections of the Heckman-Opdam theory to spherical functions on symmetric spaces in [HS] and references cited there. In the case A_{q-1}, the spherical functions on $G//K \simeq C_q^A$ are then given by

$$\varphi_\lambda^A(a_t) = e^{i \cdot \langle t - \pi(t), \lambda \rangle} \cdot F_A(i\pi(\lambda), d/2; \pi(t)) \qquad (t \in \mathbb{R}^q,\ \lambda \in \mathbb{C}^q)$$

with multiplicity $k = d/2$ where

$$\pi : \mathbb{R}^q \to \mathbb{R}_0^q := \{t \in \mathbb{R}^q : \ t_1 + \ldots + t_q = 0\}$$

is the orthogonal projection w.r.t. the standard scalar product; see e.g. Eq. (6.7) of [RKV]. In the BC-cases with $p > q$, the spherical functions on $G//K \simeq C_q^B$ are given by

$$\varphi_\lambda^p(a_t) = F_{BC}(i\lambda, k_p; t) \qquad (t \in \mathbb{R}^q,\ \lambda \in \mathbb{C}^q)$$

with three-dimensional multiplicity

$$k_p = (d(p-q)/2, (d-1)/2, d/2)$$

corresponding to the roots $\pm 2e_i$, $\pm 4e_i$ and $2(\pm e_i \pm e_j)$.

In the BC-cases, the associated double coset convolutions $*_{p,q}$ of measures on C_q^B are written down explicitly in [R2] for $p \geq 2q$ such that these convolutions and the associated product formulas for the

associated hypergeometric functions F_{BC} above can be extended to all real parameters $p \geq 2q-1$ by analytic continuation where the case $p = 2q-1$ appears as degenerated singular limit case. For these continuous family of parameters $p \in [2q-1, \infty[$, the convolutions $*_{p,q}$ are associative, commutative, and probability-preserving, and they generate commutative hypergroups $(C_q^B, *_{p,q})$ in the sense of Dunkl, Jewett, and Spector by [R2]; for the notion of hypergroups we refer to Jewett [J], where hypergroups were called convos, and to the monograph [BH]. The results of [R2] in particular imply that the (nontrivial) multiplicative continuous functions of these hypergroups $(C_q^B, *_{p,q})$ are precisely the associated hypergeometric functions $t \mapsto F_{BC}(i\lambda, k_p; t)$ with $\lambda \in \mathbb{C}^q$.

Let us now turn to a probabilistic point of view. It is well-known from probability theory on groups that G-invariant random walks on the symmetric spaces G/K as above are in a one-to-one-correspondence with random walks on the associated double coset hypergroups $(G//K, *)$ via the canonical projection from G/K onto $G//K$. In this way, all limit theorems for random walks on $(G//K, *)$ admit interpretations as limit theorems for G-invariant random walk on G/K.

The major aim of the present paper is to derive several limit theorems for time-homogeneous random walks $(X_n)_{n\geq 0}$ on the concrete double coset hypergroups $(G//K, *)$ mentioned above as well as on some generalizations. For this, we shall use an analytic approach which allows to derive all results in the BC-cases not just for the group cases $(G//K = C_q^B, *_{p,q})$ with integers p, but also for the the intermediate cases $(C_q^B, *_{p,q})$ with real numbers $p \in [2q-1, \infty[$ of Rösler [R2]. In particular we present strong laws of large numbers and central limit theorems with q-dimensional normal distributions as limits with explicit formulas for the parameters, i.e., the drift vectors and the diffusion matrices. In particular, the q-dimensional dispersion of probability measures on the Weyl-chambers C_q^A and C_q^B appears as drift depending on the concrete underlying hypergroup convolutions. For the case BC_1 of rank $q = 1$, the hypergroups $(C_q^B, *_{p,q})$ are hypergroups on $[0, \infty[$ with Jacobi functions as multiplicative functions; see [K] for the theory of Jacobi functions. These hypergroups on $[0, \infty[$ fit into the theory of non-compact one-dimensional Sturm-Liouville hypergroups, for which our approach is well-known; see [Z1], [Z2], [V1], [V2], [V3], the monograph [BH], and papers cited there.

In order to describe the dispersion and the diffusion matrices, we shall introduce analogues of multivariate moments of probability measures on C_q^A and C_q^B, which can be computed explicitly via so-called moment functions $m_{\mathbf{k}} : C_q^B \to \mathbb{R}$ for multiindices $\mathbf{k} = (k_1, \ldots, k_q) \in \mathbb{N}_0^q$ which replace the usual moment functions $x \mapsto x^{\mathbf{k}} := x_1^{k_1} \cdots x_q^{k_q}$ on the group $(\mathbb{R}^q, +)$. These moment functions $m_{\mathbf{k}}$ are defined as partial derivatives of the multiplicative functions φ_λ w.r.t. the spectral parameters at $\lambda = -i\rho$, where ρ is the half sum of positive roots, and $\varphi_{-i\rho}$ is the identity character 1 of our hypergroups on C_q^A or C_q^B.

We recall that in the group cases above, our limit theorems on the Weyl chambers C_q^A and C_q^B may be regarded as limit theorems for time-homogeneous group-invariant random walks on the associated symmetric spaces G/K for which the limit theorems of this paper are partially known for a long time; see [BL], [FH], [G1], [G2], [L], [Ri], [Te1], [Te2], [Tu], [Ri], [Vi], and references there. On the other hand, our analytic approach goes beyond the group cases in the BC-case for non-integers $p \in [2q-1, \infty[$. Moreover, we obtain explicit analytic formulas for the drift vectors and diffusion matrices below in the limit theorems which seem to be new even in the group cases.

We point out that we are interested in this paper mainly in the case BC. As the A-case in the Heckman-Opdam theory appears as a limit of the BC-case for $p \to \infty$ in some way (see [RKV], [RV1] for the details), it is not astonishing that all results in the BC-case are also available in the A-case without additional effort. In practice, all results below are proved first for the simpler A-case and then extended to the more interesting BC-case.

This paper is organized as follows. For the convenience of the reader, we collect all major results on random walks on the symmetric spaces $GL(q, \mathbb{F})/SU(q, \mathbb{F})$ and the associated Weyl chambers C_q^A of type A in Section 2 without proofs. We then do the same in Section 3 for random walks on the Grassmannian manifolds $\mathcal{G}_{p,q}(\mathbb{F})$ and the associated Weyl chambers C_q^B of type B where in the latter case the parameter $p \in [2q-1, \infty[$ is continuous. The remaining sections are then devoted to the proofs of the main results from Sections 2 and 3. In particular, in Section 4 we collect some basic results from matrix analysis which are needed later. Sections 5 and 6 contain the proofs of facts on the

moment functions in the cases A and BC respectively. There we derive some results on the uniform oscillatory behavior of the spherical functions and hypergeometric functions at the spectral parameter $-i\rho$ which may be interesting for themselves and seem to be new even for spherical functions. Sections 7 and 8 are devoted to the proofs of the laws of large numbers and central limit theorems.

We expect that at least parts of this paper may be extended from the Grassmannians $G/K = G_{p,q}(\mathbb{F})$ and the chambers C_q^B to the reductive cases $U(p,q)/(U(p)\times SU(q))$ and the space $C_q^B \times \mathbb{T}$, which may be identified with the double coset space $U(p,q)//(U(p)\times SU(q))$, and where again the spherical functions can be described in terms of the functions F_{BC}; see Ch. I.5 of [HS] and [V4].

2 Dispersion and limit theorems for root systems of type A

Consider the general linear group $G := GL(q,\mathbb{F})$ with maximal compact subgroup $K := U(q,\mathbb{F})$ with an integer $q \geq 2$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ as in the introduction. Let

$$\sigma_{sing}(g) \in \{x = (x_1,\ldots,x_q) \in \mathbb{R}^q : \; x_1 \geq x_2 \geq \cdots \geq x_q > 0\}$$

be the singular (or Lyapunov) spectrum of $g \in G$ where the singular values of g, i.e., the square roots of the eigenvalues of the positive definite matrix g^*g, are ordered by size. Using the notation $\ln(x_1,\ldots,x_q) := (\ln x_1,\ldots,\ln x_q)$, we consider the K-biinvariant mapping

$$\ln\sigma_{sing} : \; G \longrightarrow C_q^A$$

which leads to the canonical identification of the double coset space $G//K$ with the Weyl chamber C_q^A which corresponds to the identification in Eq. (1.1) in the introduction.

Now consider i.i.d. G-valued random variables $(X_k)_{k\geq 1}$ with the common K-biinvariant distribution $\nu_G \in M^1(G)$ and the associated G-valued random walk $(S_k := X_1 \cdot X_2 \cdots X_k)_{k\geq 0}$ with the convention that S_0 is the identity matrix $I_q \in G$. We now always identify the double coset space $G//K$ with C_q^A as above. Then, via taking the image measure of ν_G under the canonical projection from G to $G//K$, the K-biinvariant distribution $\nu_G \in M^1(G)$ is in a one-to-one-correspondence with some probability measure $\nu \in M^1(C_q^A)$. We shall show that, under natural moment conditions, the C_q^A-valued random variables

$$\frac{\ln\sigma_{sing}(S_k)}{k}$$

converge a.s. to some drift vector $m_{\mathbf{1}}(\nu) \in C_q^A$, and that the distributions of $\mathbb{R}^q$-valued random variables

$$\frac{1}{\sqrt{k}}(\ln\sigma_{sing}(S_k) - k\cdot m_{\mathbf{1}}(\nu)) \tag{2.1}$$

tend to some normal distribution $N(0,\Sigma^2(\nu))$ on $\mathbb{R}^q$. We shall give explicit formulas for $m_{\mathbf{1}}(\nu)$ and the covariance matrix $\Sigma^2(\nu)$ depending on ν and the dimension parameter $d = 1,2,4$ of $\mathbb{F}$.

Let us briefly compare this central limit theorem (CLT) with the existing literature. By polar decomposition of $g \in G$, the symmetric space G/K can be identified with the cone $P_q(\mathbb{F})$ of positive definite hermitian $q\times q$ matrices via

$$gK \mapsto I(g) := gg^* \in P_q(\mathbb{F}) \qquad (g \in G),$$

where G acts on $P_q(\mathbb{F})$ via $a \mapsto gag^*$. In this way, we again obtain the identification $G//K \simeq C_q^A$ via

$$KgK \mapsto \ln\sigma_{sing}(g) = \frac{1}{2}\ln\sigma(gg^*)$$

where here σ means the spectrum, i.e., the ordered eigenvalues, of a positive definite matrix. Therefore, the CLT above may be regarded as a CLT for the spectrum of K-invariant random walks on $P_q(\mathbb{F})$. CLTs in this context have a long history. In particular, [Tu], [FH], [Te1], [Te2], [Ri], [G1], and [G2] contain CLTs where, different from our CLT, ν is renormalized first into some measure $\nu_k \in M^1(G)$,

and then the convergence of the convolution powers ν_k^k is studied. Our CLT is also in principle well-known up to the explicit formulas for the drift $m_{\mathbf{1}}(\nu)$ and the covariance matrix $\Sigma^{(}\nu)$; see Theorem 1 of [Vi], the CLTs of Le Page [L], and the part of Bougerol in the monograph [BL].

We now turn to the constants $m_{\mathbf{1}}(\nu) \in C_q^A$ and $\Sigma^2(\nu)$. For this we follow the approach in [Z1], [Z2], [V1], and [BH], and introduce so-called moment functions on the double coset hypergroups $C_q^A \simeq G//K$ via partial derivatives of the spherical functions φ_λ^A w.r.t. the spectral parameter λ at the identity. For this we consider the half sum of positive roots

$$\rho = (\rho_1, \ldots, \rho_q) \qquad \text{with} \qquad \rho_l = \frac{d}{2}(q + 1 - 2l) \qquad (l = 1, \ldots, q) \tag{2.2}$$

and recapitulate the Harish-Chandra integral representation of the spherical functions

$$\varphi_\lambda^A(t) = e^{i\cdot\langle t - \pi(t), \lambda\rangle} \cdot F_A(i\pi(\lambda), d/2; \pi(t)) \qquad (t \in \mathbb{R}^q,\ \lambda \in \mathbb{C}^q) \tag{2.3}$$

from [H1], [Te2]. For this we need some notations: For a Hermitian matrix $A = (a_{ij})_{i,j=1,\ldots,q}$ over $\mathbb{F}$ we denote by $\Delta(A)$ the determinant of A, and by $\Delta_r(A) = \det((a_{ij})_{1\le i,j\le r})$ the r-th principal minor of A for $r = 1, \ldots, , q$. For $\mathbb{F} = \mathbb{H}$, all determinants are understood in the sense of Dieudonné, i.e. $\det(A) = (\det_{\mathbb{C}}(A))^{1/2}$, when A is considered as a complex matrix. For any positive Hermitian $q \times q$-matrix x and $\lambda \in \mathbb{C}^q$ we now define the power function

$$\Delta_\lambda(x) = \Delta_1(x)^{\lambda_1 - \lambda_2} \cdot \ldots \cdot \Delta_{q-1}(x)^{\lambda_{q-1} - \lambda_q} \cdot \Delta_q(x)^{\lambda_q}. \tag{2.4}$$

With these notations, the Harish-Chandra integral representation of the functions in (2.3) reads as

$$\varphi_\lambda^A(t) = \int_{U(q,\mathbb{F})} \Delta_{(i\lambda-\rho)/2}\big(u^{-1} e^{2\underline{t}} u\big)\, du; \tag{2.5}$$

see also Section 3 of [RV1] for the precise identification. It is clear from (2.5) that $\varphi_{-i\rho} \equiv 1$, and that for $\lambda \in \mathbb{R}^n$ and $t \in C_q^A$, $|\varphi_{-i\rho+\lambda}(g)| \le 1$. We mention that the set of all parameters $\lambda \in \mathbb{C}^q$, for which φ_λ is bounded, is completely known; see [R2] and [NPP].

We now follow the known approach to the dispersion for the Gelfand pairs (G, K) (see [FH], [Te1], [Te2], [Ri], [G1], [G2]) and to moment functions on hypergroups in Section 7.2.2 of [BH] (see also [Z1], [Z2], [V2], [V3]): For multiindices $l = (l_1, \ldots, l_q) \in \mathbb{N}_0^q$ we define the moment functions

$$\begin{aligned} m_l(t) :=& \frac{\partial^{|l|}}{\partial\lambda^l} \varphi_{-i\rho-i\lambda}(t)\Big|_{\lambda=0} := \frac{\partial^{|l|}}{(\partial\lambda_1)^{l_1}\cdots(\partial\lambda_n)^{l_q}} \varphi_{-i\rho-i\lambda}(t)\Big|_{\lambda=0} \\ =& \frac{1}{2^{|l|}} \int_K (\ln \Delta_1(u^{-1}e^{2\underline{t}}u))^{l_1} \cdot \left(\ln\left(\frac{\Delta_2(u^{-1}e^{2\underline{t}}u)}{\Delta_1(u^{-1}e^{2\underline{t}}u)}\right)\right)^{l_2} \cdots \left(\ln\left(\frac{\Delta_q(u^{-1}e^{2\underline{t}}u)}{\Delta_{q-1}(u^{-1}e^{2\underline{t}}u)}\right)\right)^{l_q} du \end{aligned} \tag{2.6}$$

of order $|l| := l_1 + \cdots + l_q$ for $g \in G$. Clearly, the last equality in (2.6) follows from (2.5) by interchanging integration and derivatives. Using the q moment functions of first order, we form the vector-valued moment function

$$m_{\mathbf{1}}(t) := (m_{(1,0,\ldots,0)}(t), \ldots, m_{(0,\ldots,0,1)}(t)) \tag{2.7}$$

of first order. We prove in Section 5:

2.1 Proposition. (1) *For all $t \in C_q^A$, $m_{\mathbf{1}}(t) \in C_q^A$.*

(2) *There exists a constant $C = C(q)$ such that for all $t \in C_q^A$,*

$$\|m_{\mathbf{1}}(t) - t\| \le C.$$

(3) *There exists a constant $C = C(q)$ such that for all $t \in C_q^A$ and $\lambda \in \mathbb{R}^q$,*

$$\|\varphi_{-i\rho-\lambda}^A(t) - e^{i\langle\lambda, m_{\mathbf{1}}(t)\rangle}\| \le C\|\lambda\|^2.$$

Similar to the moment function $m_{\mathbf{1}}$, we group the moment functions of second order by

$$m_{\mathbf{2}}(t) := \begin{pmatrix} m_{1,1}(t) & \cdots & m_{1,q}(t) \\ \vdots & & \vdots \\ m_{q,1}(t) & \cdots & m_{q,q}(t) \end{pmatrix} \tag{2.8}$$

$$:= \begin{pmatrix} m_{(2,0,\ldots,0)}(t) & m_{(1,1,0,\ldots,0)}(t) & \cdots & m_{(1,0,\ldots,0,1)}(t) \\ m_{(1,1,0,\ldots,0)}(t) & m_{(0,2,0,\ldots,0)}(t) & \cdots & m_{(0,1,0,\ldots,0,1)}(t) \\ \vdots & \vdots & & \vdots \\ m_{(1,0,\ldots,0,1)}(t) & m_{(0,1,0,\ldots,0,1)}(t) & \cdots & m_{(0,\ldots,0,2)}(t) \end{pmatrix} \quad \text{for} \quad t \in C_q^A.$$

We derive the following facts about the $q \times q$-matrices $\Sigma^2(t) := m_{\mathbf{2}}(t) - m_{\mathbf{1}}(t)^t \cdot m_{\mathbf{1}}(t)$ in Section 5:

2.2 Proposition. (1) *For each $t \in C_q^A$, $\Sigma^2(t)$ is positive semidefinite.*

(2) *For $t = c \cdot (1, \ldots, 1) \in C_q^A$ with $c \in \mathbb{R}$, $\Sigma^2(t) = 0$.*

(3) *If $t \in C_q^A$ does not have the form of part (2), then $\Sigma^2(t)$ has rank $q - 1$.*

(4) *For all $j, l = 1, \ldots, q$ and $t \in C_q^A$, $|m_{j,l}(t)| \le ((q-1)(t_1 - t_q) + \max(|t_1|, |t_q|))^2$.*

(5) *There exists a constant $C = C(q)$ such that for all $t \in C_q^A$,*

$$|m_{1,1}(t) - t_1^2| \le C(|t_1| + 1) \qquad \textit{and} \qquad |m_{q,q}(t) - t_q^2| \le C(|t_q| + 1).$$

By Proposition 2.2(4) and (5), all second moment functions $m_{j,l}$ are growing at most quadratically, and $m_{1,1}$ and $m_{q,q}$ are in fact growing quadratically.

Consider a probability measure $\nu \in M^1(C_q^A)$. We say that ν admits first moments if all usual first moments $\int_{C_q^A} t_j \, d\nu(t)$ $(j = 1, \ldots, q)$ exist. By Proposition 2.1(2) this is equivalent to require that the modified expectation

$$m_{\mathbf{1}}(\nu) := \int_{C_q^A} m_{\mathbf{1}}(t) \, d\nu(t) \in C_q^A \subset \mathbb{R}^q$$

exists. $m_{\mathbf{1}}(\nu)$ is called dispersion of ν. In a similar way we say that ν admits second moments if all usual second moments $\int_{C_q^A} t_j^2 \, d\nu(t)$ $(j = 1, \ldots, q)$ exist. By Proposition 2.2(4) and (5) this means that all second moment functions $m_{j,l} \ge 0$ are ν-integrable. In particular, in this case, also all moments of first order exist, and we can form the modified symmetric $q \times q$-covariance matrix

$$\Sigma^2(\nu) := \int_G m_{\mathbf{2}} \, d\nu \; - \; m_{\mathbf{1}}(\nu)^t \cdot m_{\mathbf{1}}(\nu).$$

The rank of this positive semidefinite matrix can be determined depending on ν. This follows in a natural way from the structure of the double coset hypergroup $G//K \simeq A_q^A$ which is the direct product of the diagonal subgroup $D_q := \{c \cdot (1, \ldots, 1) : \; c \in \mathbb{R}\} \subset C_q^A$ and the subhypergroup $C_q^{A,0} := \{t \in C_q^A : t_1 + \cdots + t_q = 0\}$ which is a reduced Weyl chamber of type A. This direct product structure explains the form (2.3) of the spherical functions. It also explains Proposition 2.2 and the following result on $\Sigma^2(\nu)$:

2.3 Proposition. *Assume that $\nu \in M^1(C_q^A)$ admits second moments.*

(1) *If the projection of ν under the orthogonal projection from $C_q^A \subset \mathbb{R}^q$ onto D_q is not a point measure, and if the support of ν is not contained in D_q, then $\Sigma^2(\nu)$ is positive definite.*

(2) *If $\operatorname{supp} \nu \subset D_q$, then the rank of $\Sigma^2(\nu)$ is at most 1.*

(3) *If the projection of ν under the orthogonal projection from $C_q^A \subset \mathbb{R}^q$ to D_q is a point measure, and if $\operatorname{supp} \nu \not\subset D_q$, then $\Sigma^2(\nu)$ has rank $q - 1$.*

As main results of this paper in the A-case, we have the following strong law of large numbers and CLT for a biinvariant random walk $(S_k)_{k \ge 0}$ on G associated with the probability measure $\nu \in M^1(C_q^A)$. Proofs are given in Section 7 below.

2.4 Theorem. (1) *If ν admits first moments, then for $k \to \infty$,*

$$\frac{\ln \sigma_{sing}(S_k)}{k} \longrightarrow m_{\mathbf{1}}(\nu) \qquad \textit{almost surely.}$$

(2) *If ν admits second moments, then for all $\epsilon > 1/2$ and $k \to \infty$,*

$$\frac{1}{k^\epsilon}(\ln \sigma_{sing}(S_k) - k \cdot m_{\mathbf{1}}(\nu)) \longrightarrow 0 \qquad \textit{almost surely.}$$

2.5 Theorem. *If $\nu \in M^1(G)$ admits finite second moments, then for $k \to \infty$,*

$$\frac{1}{\sqrt{k}}(\ln \sigma_{sing}(S_k) - k \cdot m_{\mathbf{1}}(\nu)) \longrightarrow N(0, \Sigma^2(\nu)) \qquad \textit{in distribution.}$$

3 Dispersion and limit theorems for root systems of type BC

In this section we consider the non-compact Grassmann manifolds $\mathcal{G}_{p,q}(\mathbb{F}) := G/K$ with $p > q$, where depending on $\mathbb{F}$, the group G is one of the indefinite orthogonal, unitary or symplectic groups $SO_0(q,p)$, $SU(q,p)$ or $Sp(q,p)$, and K the maximal compact subgroup $SO(q) \times SO(p)$, $S(U(q) \times U(p))$ or $Sp(q) \times Sp(p)$ respectively. We identify the double coset space $G//K$ with the Weyl chamber C_q^B according to Eq. (1.2). To determine the associated canonical projection from G to C_q^B, write $g \in G$ in $p \times q$-block notation as

$$g = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}$$

with $A(g) \in M_q(\mathbb{F})$, $D(g) \in M_p(\mathbb{F})$, and so on. By Eq. (1.2), the canonical projection from G to C_q^B is given by

$$g \mapsto \text{arcosh}\,(\sigma_{sing}(A(g)))$$

where σ_{sing} again denotes the ordered singular spectrum, and arcosh is taken in each component.

Similar to Section 2, we are interested in limit theorems for biinvariant random walks $(S_k := X_1 \cdot X_2 \cdots X_k)_{k \ge 0}$ on G for i.i.d. G-valued random variables $(X_k)_{k \ge 1}$ with the common K-biinvariant distribution $\nu_G \in M^1(G)$. We identify $G//K$ with C_q^B as above. Then, via taking the image measure of ν_G under the canonical projection from G to $G//K$, the K-biinvariant distribution $\nu_G \in M^1(G)$ corresponds with some unique probability measure $\nu \in M^1(C_q^B)$. We shall show that, under natural moment conditions, the C_q^B-valued random variables

$$\frac{\text{arcosh}\,(\sigma_{sing}(A(S_k)))}{k}$$

converge a.s. to some drift vector $m_{\mathbf{1}}(\nu) \in C_q^B$, and that the $\mathbb{R}^q$-valued random variables

$$\frac{1}{\sqrt{k}}(\text{arcosh}\,(\sigma_{sing}(A(S_k))) - k \cdot m_{\mathbf{1}}(\nu)) \tag{3.1}$$

tend in distribution to some normal distribution $N(0, \Sigma^2(\nu))$ on $\mathbb{R}^q$.

We derive these limit theorems in a more general context. For this recall that $C_q^B \simeq G//K$ is a double coset hypergroup whose multiplicative functions are given by the hypergeometric functions

$$\varphi_\lambda^p(t) = F_{BC}(i\lambda, k_p; t) \qquad (t \in C_q^B,\ \lambda \in \mathbb{C}^q) \tag{3.2}$$

with multiplicity $k_p = (d(p-q)/2, (d-1)/2, d/2)$. In [R2], the product formula for these spherical functions $\varphi \in C(G)$, namely

$$\varphi(g)\varphi(h) = \int_K \varphi(gkh)\, dk \qquad (g, h \in G),$$

was written down explicitly in terms of these hypergeometric functions of type BC for all $p \geq 2q$ as a product formula for on $G//K \simeq C_q^B$ such that this formula remains correct for φ_λ^p with all real parameters $p \in]2q-1,\infty]$. This result from [R2] is as follows: For all $s,t \in C_q^B$ and $\lambda \in \mathbb{C}^q$,

$$\varphi_\lambda^p(t)\varphi_\lambda^p(s) = \int_{C_q^B} \varphi_\lambda^p(x)\, d(\delta_s *_p \delta_t)(x)$$

where the probability measures $\delta_s *_p \delta_t \in M^1(C_q^B)$ with compact support are given by

$$(\delta_s *_p \delta_t)(f) = \frac{1}{\kappa_p}\int_{B_q}\int_{U(q,\mathbb{F})} f\Big(\operatorname{arcosh}\left(\sigma_{sing}(\sinh \underline{t}\, w \,\sinh \underline{s} \,+\, \cosh \underline{t}\, v\, \cosh \underline{s})\right)\Big)\, dv\, dm_p(w) \quad (3.3)$$

for functions $f \in C(C_q^B)$. Here, dv means integration w.r.t. the normalized Haar measure on $U(q,\mathbb{F})$, B_q is the matrix ball

$$B_q := \{w \in M_q(\mathbb{F}) :\ w^*w \leq I_q\},$$

and $dm_p(w)$ is the probability measure

$$dm_p(w) := \frac{1}{\kappa_p}\Delta(I - w^*w)^{d(p/2+1/2-q)-1}\, dw \quad \in M^1(B_q) \quad (3.4)$$

where dw is the Lebesgue measure on the ball B_q, and the normalization constant $\kappa_p > 0$ is chosen such that $dm_p(w)$ is a probability measure. For $p = 2q-1$ there is a corresponding degenerated formula where then $m_p \in M^1(B_q)$ then becomes singular; see Section 3 of [R1] for details. By [R2], the convolution (3.3) can be extended for all $p \in [2q-1,\infty[$ in a unique bilinear, weakly continuous way to a commutative and associative convolution $*_p$ on the Banach space of all bounded Borel measures on C_q^B, such that $(C_q^B, *_p)$ becomes a commutative hypergroup with $0 \in \mathbb{R}^q$ as identity.

We now use the convolution $*_p$ for $p \in [2q-1,\infty[$ and $d = 1,2,4$ and generalize the Markov processes

$$\big(\tilde{S}_k := \operatorname{arcosh}\left(\sigma_{sing}(A(S_k))\right)\big)_{k\geq 0} \qquad \text{on} \quad C_q^B \quad (3.5)$$

in the group cases for integers p as follows: Fix $\nu \in M^1(C_q^B)$, and consider a time-homogeneous random walk $(\tilde{S}_k)_{k\geq 0}$ on C_q^B (associated with the parameters p,d) with law ν, i.e., a time-homogeneous Markov process on starting at the hypergroup identity $0 \in C_q^B$ with transition probability

$$P(\tilde{S}_{k+1} \in A |\, \tilde{S}_k = x) = (\delta_x * \nu)(A) \qquad (x \in C_q^B,\ A \subset C_q^B \quad \text{a Borel set}).$$

By our construction, each stochastic process on C_q^B defined via Eq. (3.5), is in fact such a time-homogeneous random walk for the corresponding p,d. We also point out that induction on k shows easily that the distributions of $\tilde{S}_k$ are given as the convolution powers $\nu^{(k)}$ w.r.t. the convolution $*_p$. We shall derive all limit theorems in this setting for $p \in]2q-1,\infty[$.

To identify the data of the limit theorems, we proceed as in Section 2 and use the Harish-Chandra integral representation of φ_λ^p in Theorem 2.4 of [RV1]:

3.1 Proposition. *For all $p > 2q-1$, $t \in C_q^B$, and $\lambda \in \mathbb{C}^q$,*

$$\varphi_\lambda^p(t) = \int_{B_q}\int_{U(q,\mathbb{F})} \Delta_{(i\lambda-\rho^{BC})/2}(g(t,u,w))\, du\, dm_p(w) \quad (3.6)$$

with the power function Δ_λ from (2.4), the half sum of positive roots

$$\rho^{BC} = \rho^{BC}(p) = \sum_{i=1}^q \Big(\frac{d}{2}(p+q+2-2i)-1\Big)e_i\,, \quad (3.7)$$

$$g(t,u,w) := u^*(\cosh \underline{t} + \sinh \underline{t}\cdot w)(\cosh \underline{t} + \sinh \underline{t}\cdot w)^* u, \quad (3.8)$$

and with $m_p(w) \in M^1(B_q)$ from (3.4). For $p = 2q-1$, a corresponding degenerated formula holds.

Proof. This formula follows immediately from Theorem 2.4 of [RV1]. Notice that that our function $g(t,u,w)$ is equal to the function $\tilde{g}_t(u,w)$ in Section 2 of [RV1]. Moreover, in [RV1] we take one integral over the identity component $U_0(q,\mathbb{F})$ of $U(q,\mathbb{F})$ instead over $U(q,\mathbb{F})$. But this makes a difference for these groups for $\mathbb{F}=\mathbb{R}$ only, where the integrals are equal in all cases by the form of $g(t,u,w)$. $\square$

We now proceed as in Section 2. For $l=(l_1,\ldots,l_q)\in\mathbb{N}_0^q$ we define the moment functions

$$\begin{aligned} m_l(t) &:= \frac{\partial^{|l|}}{\partial\lambda^l}\varphi^p_{-i\rho^{BC}-i\lambda}(t)\Big|_{\lambda=0} := \frac{\partial^{|l|}}{(\partial\lambda_1)^{l_1}\cdots(\partial\lambda_q)^{l_q}}\varphi^p_{-i\rho^{BC}-i\lambda}(t)\Big|_{\lambda=0} \\ &= \frac{1}{2^{|l|}}\int_{B_q}\int_{U(q,\mathbb{F})}(\ln\Delta_1(g(t,u,w)))^{l_1}\cdot\left(\ln\frac{\Delta_2(g(t,u,w))}{\Delta_1(g(t,u,w))}\right)^{l_2}\cdots\left(\ln\frac{\Delta_q(g(t,u,w))}{\Delta_{q-1}(g(t,u,w))}\right)^{l_q} du\, dm_p(w) \end{aligned} \tag{3.9}$$

of order $|l|$ for $t\in C_q^B$. Clearly, the last equality follows from (3.6) by interchanging integration and derivatives. Using the q moment functions m_l of first order with $|l|=1$, we form the vector-valued moment function

$$m_{\mathbf{1}}(t) := (m_{(1,0,\ldots,0)}(t),\ldots,m_{(0,\ldots,0,1)}(t)) \tag{3.10}$$

of first order. We prove the following properties of $m_{\mathbf{1}}$ in Section 6:

3.2 Proposition. (1) *For all $t\in C_q^B$, $m_{\mathbf{1}}(t)\in C_q^B$.*

(2) *There exists a constant $C=C(p,q)$ such that for all $t\in C_q^B$,*

$$\|m_{\mathbf{1}}(t)-t\|\le C.$$

(3) *There exists a constant $C=C(p,q)$ such that for all $t\in C_q^B$ and $\lambda\in\mathbb{R}^q$,*

$$\|\varphi^A_{-i\rho-\lambda}(t)-e^{i\langle\lambda,m_{\mathbf{1}}(t)\rangle}\|\le C\|\lambda\|^2.$$

As in Section 2 we also form the matrix consisting of all second order moment functions with

$$m_{\mathbf{2}}(t) := \begin{pmatrix} m_{1,1}(t) & \cdots & m_{1,q}(t) \\ \vdots & & \vdots \\ m_{q,1}(t) & \cdots & m_{q,q}(t)\end{pmatrix} \tag{3.11}$$

$$:= \begin{pmatrix} m_{(2,0,\ldots,0)}(t) & m_{(1,1,0,\ldots,0)}(t) & \cdots & m_{(1,0,\ldots,0,1)}(t) \\ m_{(1,1,0,\ldots,0)}(t) & m_{(0,2,0,\ldots,0)}(t) & \cdots & m_{(0,1,0,\ldots,0,1)}(t) \\ \vdots & \vdots & & \vdots \\ m_{(1,0,\ldots,0,1)}(t) & m_{(0,1,0,\ldots,0,1)}(t) & \cdots & m_{(0,\ldots,0,2)}(t)\end{pmatrix} \quad \text{for} \quad t\in C_q^B.$$

By Section 6, the symmetric $q\times q$-matrices $\Sigma^2(t):=m_{\mathbf{2}}(t)-m_{\mathbf{1}}(t)^t\cdot m_{\mathbf{1}}(t)$ have the following properties:

3.3 Proposition. (1) *For each $t\in C_q^B$, the matrix $\Sigma^2(t)$ is positive semidefinite.*

(2) $\Sigma^2(0)=0$.

(3) *For $t\in C_q^B$ with $t\neq 0$, the matrix $\Sigma^2(t)$ has full rank q.*

(4) *There exists a constant $C=C(q)$ such that for all $j,l=1,\ldots,q$ and $t\in C_q^B$, $|m_{j,l}(t)|\le C\cdot t_1^2$.*

(5) *There exists a constant $C=C(q)$ such that for all $t\in C_q^B$,*

$$|m_{1,1}(t)-t_1^2|\le C(|t_1|+1).$$

Parts (4), (5) yield that all second moment functions $m_{j,l}$ are growing at most quadratically, and that at least $m_{1,1}$ is growing quadratically.

Now consider a probability measure $\nu\in M^1(C_q^B)$. As in Section 2 we say that ν admits first or second moments if all components of $m_{\mathbf{1}}$ or $m_{\mathbf{2}}$ are integrable w.r.t. ν respectively. In case of existence, we form the vector $m_{\mathbf{1}}(\nu)\in C_q^B$ and the matrix $\Sigma^2(\nu)$ as in Section 2. We then have the following result which is slightly different from the corresponding one in the A-case in Section 2:

3.4 Proposition. *If $\nu \in M^1(C_q^B)$ admits second moments, and if $\nu \neq \delta_0$, then $\Sigma^2(\nu)$ has full rank q.*

As main results of this paper in the BC-case, we have the following strong law of large numbers and CLT for time-homogeneous random walk $(\tilde{S}_k)_{k\geq 0}$ on G associated with the probability measure $\nu \in M^1(C_q^B)$ which is completely analog to the corresponding results in the A-case in Section 2. The proofs, which are completely analog to the A-case, are given in Section 8.

3.5 Theorem. (1) *If ν admits first moments, then for $k \to \infty$,*

$$\frac{\tilde{S}_k}{k} \longrightarrow m_{\mathbf{1}}(\nu) \qquad a.s..$$

(2) *If ν admits second moments, then for all $\epsilon > 1/2$ and $k \to \infty$,*

$$\frac{1}{k^\epsilon}(\tilde{S}_k - k \cdot m_{\mathbf{1}}(\nu)) \longrightarrow 0 \qquad \textit{almost surely.}$$

3.6 Theorem. *If $\nu \in M^1(G)$ admits finite second moments, then for $k \to \infty$,*

$$\frac{1}{\sqrt{k}}(\tilde{S}_k - k \cdot m_{\mathbf{1}}(\nu)) \longrightarrow N(0, \Sigma^2(\nu)) \qquad \textit{in distribution.}$$

4 Some results from matrix analysis

It this section we collect some results from matrix analysis which are needed later. Possibly, some of these results are well-known, but we were unable to find references. We always assume that $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $q \geq 2$. Moreover, $M_r(\mathbb{F})$ is the vector space of all $r \times r$-matrices over $\mathbb{F}$.

We start with the following observation from linear algebra.

4.1 Lemma. *Let $u \in U(q, \mathbb{F})$ have the block structure $u = \begin{pmatrix} u_1 & * \\ * & u_2 \end{pmatrix}$ with quadratic blocks $u_1 \in M_r(\mathbb{F})$ and $u_2 \in M_{q-r}(\mathbb{F})$ with $1 \leq r \leq q$. Then $|\det u_1| = |\det u_2|$.*

Proof. W.l.o.g. we assume $2r \leq q$. By the KAK-decomposition of $U(q, \mathbb{F})$ with $K = U(r, \mathbb{F}) \times U(q-r, \mathbb{F})$ (see e.g. Theorem VII.8.6 of [H2]), we may write u as

$$u = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \cdot \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & I_{q-2r} \end{pmatrix} \cdot \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$$

with $a_1, a_2 \in U(r, \mathbb{F})$, $b_1, b_2 \in U(q-r, \mathbb{F})$ and

$$c = diag(\cos\varphi_1, \ldots, \cos\varphi_r), \qquad s = diag(\sin\varphi_1, \ldots, \sin\varphi_r)$$

for suitable $\varphi_1, \ldots, \varphi_r \in \mathbb{R}$. Therefore,

$$u_1 = a_1 c a_2 \qquad \text{and} \qquad u_2 = b_1 \begin{pmatrix} c & 0 \\ 0 & I_{q-2r} \end{pmatrix} b_2$$

which immediately implies the claim. □

We next turn to some results on the principal minors Δ_r:

4.2 Lemma. *Let $1 \leq r \leq q$ be integers and $u \in U(q, \mathbb{F})$. Consider the polynomial*

$$h_r(a_1, \ldots, a_q) := \Delta_r(u^* \cdot diag(a_1, \ldots, a_q) \cdot u) \qquad \textit{for} \quad a_1, \ldots, a_q \in \mathbb{R}.$$

Then

$$h_r(a_1, \ldots, a_q) = \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq q} c_{i_1,\ldots,i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r}$$

with coefficients $c_{i_1,\ldots,i_r} \geq 0$ for all $1 \leq i_1 < i_2 < \ldots < i_r \leq q$ and $\sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq q} c_{i_1,\ldots,i_r} = 1$.

Proof. Clearly, h_r is homogeneous of degree r, i.e.,

$$h_r(a_1, \ldots, a_q) = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_r \leq q} c_{i_1,\ldots,i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r}.$$

We first check that $c_{i_1,\ldots,i_r} \neq 0$ is possible only for coefficients with $1 \leq i_1 < i_2 < \ldots < i_r \leq q$. For this consider indices $i_1, \ldots, i_r$ with $|\{i_1, \ldots, i_r\}| =: n < r$. By changing the numbering of the variables $a_1, \ldots, a_q$ (and of rows and columns of u in an appropriate way), we may assume that $\{i_1, \ldots, i_r\} = \{1, \ldots, n\}$. In this case, $u^* \cdot diag(a_1, \ldots, a_n, 0, \ldots, 0) \cdot u$ has rank at most $n < r$. Thus

$$0 = h_r(a_1, \ldots, a_n, 0, \ldots, 0) = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_r \leq n} c_{i_1,\ldots,i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r}$$

for all $a_1, \ldots, a_n \in \mathbb{R}$. This yields $c_{i_1,\ldots,i_r} = 0$ for $1 \leq i_1 \leq i_2 \leq \ldots \leq i_r \leq n$. Therefore, for suitable coefficients,

$$h_r(a_1, \ldots, a_q) = \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq q} c_{i_1,\ldots,i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r}.$$

For the nonnegativity we again may restrict our attention to the coefficient $c_{1,\ldots,r}$. In this case, with respect to the usual ordering of positive definite matrices,

$$0 \leq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \leq I_q \qquad \text{and thus} \qquad 0 \leq u^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} u \leq I_q.$$

As this inequality holds also for the upper left $r \times r$ block,

$$c_{1,\ldots,r} = h_r(1, \ldots, 1, 0, \ldots, 0) = \Delta_r \left(u^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} u \right) \geq 0.$$

Finally, as

$$\sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq q} c_{i_1,\ldots,i_r} = h_r(1, \ldots, 1) = 1,$$

the proof is complete. □

Let us keep the notation of Lemma 4.2. We compare h_r with the homogeneous polynomial

$$C_r(a_1, \ldots, a_q) := \frac{1}{\binom{q}{r}} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq q} a_{i_1} a_{i_2} \cdots a_{i_r} > 0 \qquad (r = 1, \ldots, q). \tag{4.1}$$

4.3 Lemma. *For all $a_1, \ldots, a_q > 0$,*

$$0 < \frac{C_r(a_1, \ldots, a_q)}{h_r(a_1, \ldots, a_q)} \leq \frac{1}{\binom{q}{r}} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq q} c_{i_1,\ldots,i_r}(u)^{-1},$$

where, depending on u, on both sides the value ∞ is possible.

Proof. Positivity is clear by Lemma 4.2. Moreover,

$$\begin{aligned} C_r(a_1, \ldots, a_q) &= \frac{1}{\binom{q}{r}} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq q} a_{i_1} a_{i_2} \cdots a_{i_r} \\ &\leq \frac{\max_{1 \leq i_1 < i_2 < \ldots < i_r \leq q} c_{i_1,\ldots,i_r}^{-1}}{\binom{q}{r}} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq q} c_{i_1,\ldots,i_r} a_{i_1} a_{i_2} \cdots a_{i_r} \end{aligned}$$

which immediately leads to the claim. □

We shall also need an integrability result for principal minors of matrices $k \in K := U(q, \mathbb{F})$. For this, we write k as block matrix $k = \begin{pmatrix} k_r & * \\ * & k_{q-r} \end{pmatrix}$ with $k_r \in M_r(\mathbb{C})$ and $k_{q-r} \in M_{q-r}(\mathbb{C})$.

4.4 Proposition. *Keep the block matrix notation above. For $0 \le \epsilon < 1/2$,*

$$\int_K |\det k_r|^{-2\epsilon}\, dk < \infty.$$

Proof. The statement is clear for $r = q$. By Lemma 4.1 we may also assume $1 \le r \le q/2$. In this case, we use the matrix ball

$$B_r := \{w \in M_r(\mathbb{F}) : \; w^*w \le I_r\}$$

as well as the ball $B := \{y \in M_{1,r}(\mathbb{F}) \equiv \mathbb{F}^r : \; \|y\|_2^2 \le 1\}$. We conclude from the truncation lemma 2.1 of [R2] that

$$\int_K |\det k_r|^{-2\epsilon}\, dk = \frac{1}{\kappa_r} \int_{B_r} |\det w|^{-2\epsilon} \Delta(I_r - w^*w)^{(q-2r+1)\cdot d/2-1}\, dw$$

where dw is the Lebesgue measure on the ball B_r and

$$\kappa_r := \int_{B_r} \det(I_r - w^*w)^{(q-2r+1)\cdot d/2-1}\, dw \in]0, \infty[.$$

Moreover, by Lemma 3.7 and Corollary 3.8 of [R1], the mapping $P : B^r \to B_r$ with

$$P(y_1, \ldots, y_r) := \begin{pmatrix} y_1 \\ y_2(I_r - y_1^*y_1)^{1/2} \\ \vdots \\ y_r(I_r - y_{r-1}^*y_{r-1})^{1/2} \cdots (I_r - y_1^*y_1)^{1/2} \end{pmatrix} \tag{4.2}$$

establishes a diffeomorphism such that the image of the measure $\det(I_r - w^*w)^{(q-2r+1)\cdot d/2-1}dw$ under P^{-1} is $\prod_{j=1}^r (1 - \|y_j\|_2^2)^{(q-r-j+1)\cdot d/2-1} dy_1 \ldots dy_r$. Moreover, we show in Lemma 4.5 below that

$$\det P(y_1, \ldots, y_r) = \det \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}.$$

We thus conclude that

$$\int_K |\det k_r|^{-2\epsilon}\, dk = \frac{1}{\kappa_r} \int_B \cdots \int_B \left| \det \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \right|^{-2\epsilon} \prod_{j=1}^r (1 - \|y_j\|_2^2)^{(q-r-j+1)\cdot d/2-1} dy_1 \ldots dy_r. \tag{4.3}$$

This integral is finite for $\epsilon < 1/2$, as one can use Fubini with an one-dimensional inner integral w.r.t. the (1,1)-variable. After this inner integration, no further singularities appear from the determinant-part in the remaining integral. $\square$

4.5 Lemma. *Keep the notations of the preceding proof. For all $y_1, \ldots, y_r \in B$,*

$$\det P(y_1, \ldots, y_r) = \det \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}.$$

Proof. Fix $y_1 \in B$. The mapping $y \mapsto y(I_r - y_1^* y_1)^{1/2}$ on B has the following form: If y is written as $y = ay_1 + y^\perp$ in a unique way with $a \in \mathbb{F}$ and $y^\perp \perp y_1$, then $y(I_r - y_1^* y_1)^{1/2} = \sqrt{1 - \|y_1\|_2^2} \cdot ay_1 + y^\perp$ (write $I_r - y_1^* y_1$ in an orthonormal basis with $y_1/\|y_1\|_2$ as a member!). Using linearity of the determinant in all lines, we thus conclude that

$$\det \begin{pmatrix} y_1 \\ y_2(I_r - y_1^* y_1)^{1/2} \\ \vdots \\ y_r(I_r - y_{r-1}^* y_{r-1})^{1/2} \cdots (I_r - y_1^* y_1)^{1/2} \end{pmatrix} = \det \begin{pmatrix} y_1 \\ y_2 \\ y_3(I_r - y_2^* y_2)^{1/2} \\ \vdots \\ y_r(I_r - y_{r-1}^* y_{r-1})^{1/2} \cdots (I_r - y_2^* y_2)^{1/2} \end{pmatrix}.$$

The lemma now follows by an obvious induction. □

In the end of this section we present a technical result which will be central below to derive that the covariance matrices Σ of Sections 2 and 3 have maximal rank in the non-degenerated cases.

4.6 Lemma. *Let $a_1, \ldots, a_q \in]0, \infty[$ such that at least two of these numbers are different. Consider the diagonal matrix $a = diag(a_1, \ldots, a_q)$. Then the functions*

$$f_r : U(q, \mathbb{F}) \to \mathbb{R}, \quad k \mapsto \ln \Delta_r(k^* ak)$$

with $r = 1, \ldots, q-1$ and the constant function 1 on $U(q, \mathbb{F})$ are linearly independent.

Proof. Without loss of generality we assume that $a = diag(a_1, \ldots, a_1, a_{s+1}, \ldots, a_q)$ where a_1 appears s-times with $1 \le s \le q-1$, and where a_1 is different from $a_{s+1}, \ldots, a_q$.

We consider permutation matrices $k_1, \ldots, k_q \in U(q, \mathbb{F})$ as follows: k_1 is the identity. For $l = 2, \ldots, s$, the matrix $k_l^* ak_l$ appears from a by interchanging the entries $l-1$ and $s+1$, and finally, for $l = s+1, \ldots, q$, the matrix $k_l^* ak_l$ appears from a by interchanging the entries s and l.

We now form the $q \times q$-dimensional matrix A with entries

$$A_{r,l} := \begin{cases} \ln \Delta_r(k_l^* ak_l) & \text{for} \quad r = 1, \ldots, q-1 \\ 1 & \text{for} \quad r = q. \end{cases}$$

and check that A is nonsingular which implies the claim.

For this we use the abbreviations $x := \ln a_1$ and $y_l := \ln a_{s+l}$ for $l = 1, \ldots, q-s$. We now subtract the first column of A from all other columns of A and obtain the matrix

$$\left(\begin{array}{c|ccccc|cccc}
x & y_1 - x & 0 & 0 & \cdots & 0 & & & & \\
2x & y_1 - x & y_1 - x & & & & & & & \\
\vdots & \vdots & \vdots & \ddots & \mathbf{0} & & & \mathbf{0} & & \\
(s-1)x & y_1 - x & y_1 - x & \cdots & y_1 - x & 0 & 0 & \cdots & 0 & 0 \\
sx & y_1 - x & y_1 - x & \cdots & y_1 - x & y_1 - x & y_2 - x & \cdots & y_{q-s-1} - x & y_{q-s} - x \\
\hline
sx + y_1 & 0 & 0 & \cdots & 0 & 0 & y_2 - x & \cdots & y_{q-s-1} - x & y_{q-s} - x \\
sx + y_1 + y_2 & 0 & 0 & \cdots & 0 & 0 & 0 & \ddots & \vdots & y_{q-s} - x \\
\vdots & & & \mathbf{0} & & & \vdots & \mathbf{0} & \ddots & \vdots \\
sx + \sum_{l=1}^{s-q-1} y_l & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & y_{q-s} - x \\
\hline
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right)$$

Using the block structure of B and the triangular form of these blocks we see that this matrix has the determinant

$$1 \cdot (y_2 - x)(y_3 - x) \cdots (y_{q-s} - x)(y_1 - x)^s \neq 0.$$

Therefore, this matrix and thus A are nonsingular as claimed. □

5 Oscillatory behavior of hypergeometric functions of type A at the identity

In this section we prove Propositions 2.1, 2.2, and 2.3 about the moment functions of first and second order in Section 2. The most remarkable result in our eyes is the oscillatory behavior of hypergeometric functions of type A at the identity character in Proposition 2.1(3) which is uniform for $t \in C_q^A$.

The proof of this fact relies on the results in Section 4 and on the following elementary observation:

5.1 Lemma. *Let $\epsilon \in]0,1]$, $M \geq 1$ and $m \in \mathbb{N}$. Then there exists a constant $C = C(\epsilon, M, m) > 0$ such that for all $z \in]0, M]$,*

$$|\ln(z)|^m \leq C\left(1 + z^{-\epsilon}\right).$$

Proof. Elementary calculus yields $|x^\epsilon \cdot \ln x| \leq 1/(e\epsilon)$ for $x \in]0,1]$ and the Euler number $e = 2,71...$. This leads to the estimate for $z \in]0,1]$. The estimate is trivial for $z \in]1, M]$. □

Proof of Proposition 2.1(3): Let $\lambda \in \mathbb{R}^q$, $t \in C_q^A$. Consider

$$a := (a_1, \ldots, a_q) := (e^{2t_1}, \ldots, e^{2t_q}), \quad \text{and} \quad a_t^2 = e^{2\underline{t}} = diag(a_1, \ldots, a_q) \in GL(q, \mathbb{F}).$$

Then, by the Harish-Chandra integral representation (2.5) and the integral representation of the moment functions in (2.6), we have to estimate

$$\begin{aligned} R := &R(\lambda, t) := |\varphi^A_{-i\rho-\lambda}(t) - e^{i\langle\lambda, m_{\mathbf{1}}(t)\rangle}| \\ = &\Bigg|\int_K exp\left(\frac{i}{2}\sum_{r=1}^q(\lambda_r - \lambda_{r+1}) \cdot \ln \Delta_r(k^* a_t^2 k)\right) dk \\ &\quad - exp\left(\frac{i}{2}\int_K \sum_{r=1}^q (\lambda_r - \lambda_{r+1}) \cdot \ln \Delta_r(k^* a_t^2 k)\, dk\right)\Bigg| \end{aligned} \tag{5.1}$$

with the convention $\lambda_{q+1} := 0$. For $r = 1, \ldots, q$, we now use the polynomial C_r from Eq. (4.1) and write the logarithms of the principal minors in (5.1) as

$$\ln \Delta_r(k^* a_t^2 k) = \ln C_r(a_1, \ldots, a_r) + \ln(H_r(k,a)) \quad \text{with} \quad H_r(k,a) := \frac{\Delta_r(k^* a_t^2 k)}{C_r(a_1, \ldots, a_n)}. \tag{5.2}$$

With this notation and with $|e^{ix}| = 1$ for $x \in \mathbb{R}$, we rewrite (5.1) as

$$\begin{aligned} R = &\Bigg|\int_K exp\left(\frac{i}{2}\sum_{r=1}^q(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k,a))\right) dk \\ &\quad - exp\left(\frac{i}{2}\int_K \sum_{r=1}^q (\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k,a))\, dk\right)\Bigg|. \end{aligned} \tag{5.3}$$

We now use the power series for both exponential functions where the terms of order 0 and 1 are equal. Hence, $R \leq R_1 + R_2$ for

$$R_1 := \int_K \left| exp\left(\frac{i}{2}\sum_{r=1}^q(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k,a))\right) - \left(1 + \frac{i}{2}\sum_{r=1}^q(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k,a))\right)\right| dk,$$

$$R_2 := \left| exp\left(\frac{i}{2}\int_K \sum_{r=1}^q(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k,a))\, dk\right) - 1 - \frac{i}{2}\int_K \sum_{r=1}^q(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k,a))\, dk\right|.$$

Using the well-known elementary estimates $|\cos x - 1| \leq x^2/2$ and $|\sin x - x| \leq x^2/2$ for $x \in \mathbb{R}$, we obtain $|e^{ix} - (1 + ix)| \leq x^2$ for $x \in \mathbb{R}$. Therefore, defining

$$A_m := 2^{-m}\int_K \left|\sum_{r=1}^q(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(k,a))\right|^m dk \qquad (m = 1, 2),$$

we conclude that

$$R \le R_1 + R_2 \le A_2 + A_1^2.$$

In the following, let $D_1, D_2, \ldots$ suitable constants. As $A_1^2 \le A_2$ by Jensen's inequality, and as

$$A_2 \le \|\lambda\|^2 \cdot D_1 \cdot \int_K \sum_{r=1}^{q} |\ln(H_r(k,a))|^2 \, dk =: \|\lambda\|^2 \cdot B_2,$$

we obtain $R \le B_2 \cdot 2\|\lambda\|^2$. To complete the proof, we must check that B_2, i.e., the integrals

$$L_r := \int_K |\ln(H_r(k,a))|^2 \, dk \tag{5.4}$$

remain bounded independent of $a_1, \ldots, a_q > 0$ for $r = 1, \ldots, q$.

For this fix r. Lemma 4.2 in particular implies that for all $a_1, \ldots, a_q > 0$,

$$\Delta_r(k^* a_t^2 k) \le \sum_{1 \le i_1 < i_2 < \ldots < i_r \le q} a_{i_1} \cdot a_{i_2} \cdots a_{i_r} = \binom{q}{r} C_r(a_1, \ldots, a_q)$$

and $\Delta_r(k^* a_t^2 k) > 0$. Hence,

$$0 < \frac{\Delta_r(k^* a_t^2 k)}{C_r(a_1, \ldots, a_q)} = H_r(k,a) \le \binom{q}{r}. \tag{5.5}$$

We conclude from (5.4), (5.5) and Lemma 5.1 that for any $\epsilon \in]0,1[$ and suitable $D_2 = D_2(\epsilon)$,

$$L_r \le D_2 \int_K \left(1 + H_r(a_1, \ldots, a_q)^{-\epsilon}\right) \, dk.$$

Thus, by Lemma 4.3,

$$\begin{aligned} L_r \le & D_2 + D_3 \int_K \left(\sum_{1 \le i_1 < i_2 < \ldots < i_r \le q} c_{i_1,\ldots,i_r}(k)^{-1} \right)^{\epsilon} dk \\ \le & D_2 + D_3 \cdot \binom{q}{r}^{\epsilon} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le q} \int_K c_{i_1,\ldots,i_r}(k)^{-\epsilon} \, dk. \end{aligned} \tag{5.6}$$

The right hand side of (5.6) is independent of $a_1, \ldots, a_q$, and, by the definition of the $c_{i_1,\ldots,i_r}(k)$ in Lemma 4.2, $\int_K c_{i_1,\ldots,i_r}(k)^{-\epsilon} \, dk$ is independent of $1 \le i_1 < i_2 < \ldots < i_r \le q$. Therefore, it suffices to check that

$$I_r := \int_K c_{1,\ldots,r}(k)^{-\epsilon} \, dk = \int_K \Delta_r \left(k^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} k \right)^{-\epsilon} dk < \infty. \tag{5.7}$$

For this, we write k as block matrix $k = \begin{pmatrix} k_r & * \\ * & k_{q-r} \end{pmatrix}$ with $k_r \in M_r(\mathbb{C})$ and $k_{q-r} \in M_{q-r}(\mathbb{C})$ and observe that

$$\Delta_r \left(k^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} k \right) = \Delta_r \begin{pmatrix} k_r^* k_r & * \\ * & * \end{pmatrix} = |\det k_r|^2.$$

Therefore, (5.7) follows from Proposition 4.4, which completes the proof of Proposition 2.1(3). □

We now turn to the proof of the remaining parts of Proposition 2.1. Part (1) is a direct consequence of the law of large numbers 2.4(1). Notice that in fact the proof of this law of large numbers in Section 7 does not depend on Proposition 2.1(1). Proposition 2.1(2) is just part (3) of the following result:

5.2 Lemma. *For $r = 1, \ldots, q$ let*

$$s_r(t) := m_{(1,0,\ldots,0)}(t) + \cdots + m_{(0,\ldots,0,1,0,\ldots,0)}(t) \qquad \textit{for} \quad t \in C_q^A$$

be the sum of the first r moment functions of first order. Then:

(1) *For all* $t \in C_q^A$, $s_q(t) = t_1 + t_2 + \cdots + t_q$.

(2) *There is a constant* $C = C(q)$ *such that for all* $r = 1, \ldots, q$ *and* $t \in C_q^A$,

$$0 \le t_1 + t_2 + \cdots + t_r \; - \; s_r(t) \le C.$$

(3) *There is a constant* $C = C(q)$ *such that for all* $t \in C_q^A$

$$\|t - m_{\mathbf{1}}(t)\| \le C.$$

Proof. By the integral representation (2.6) of the moment functions, we have

$$s_r(t) = \frac{1}{2} \int_K \ln \Delta_r(k^* e^{2\underline{t}} k)\, dk \qquad (r = 1, \ldots, q). \tag{5.8}$$

For $r = q$, this proves (1). Moreover, for $t \in C_q^A$ we have $t_1 \ge t_2 \ge \ldots \ge t_q$. This and Lemma 4.2 imply that for all $k \in K$,

$$\frac{1}{2} \ln \Delta_r(k^* e^{2\underline{t}} k) \le t_1 + t_2 + \cdots + t_r. \tag{5.9}$$

This and (5.8) now lead to the first inequality of (2). For the second inequality of (2), we use the notations of Lemmas 4.2 and 4.3. For $k \in K$ and $a_1 := e^{2t_1} \ge a_2 := e^{2t_2} \ge \ldots \ge a_q := e^{2t_q}$ we obtain from Lemma 4.3 that

$$a_1 \cdot a_2 \cdots a_r \le \binom{q}{r} C_r(a_1, \ldots, a_q) \le \Delta_r(k^* e^{2\underline{t}} k) \cdot M(k)$$

with

$$M(k) := \max_{1 \le i_1 < \ldots < i_r \le q} c_{i_1, \ldots, i_r}(k)^{-1}$$

which may be equal to ∞ for some k. Therefore,

$$\begin{aligned} t_1 + t_2 + \cdots + t_r &= \frac{1}{2} \ln(a_1 \cdot a_2 \cdots a_r) = \frac{1}{2} \int_K \ln(a_1 \cdot a_2 \cdots a_r)\, dk \\ &\le \frac{1}{2} \int_K \ln \Delta_r(k^* e^{2\underline{t}} k)\, dk + \int_K \ln M(k)\, dk \end{aligned} \tag{5.10}$$

with

$$\int_K \ln M(k)\, dk \le M := \sum_{1 \le i_1 < \ldots < i_r \le q} \int_K \ln(c_{i_1, \ldots, i_r}(k)^{-1})\, dk.$$

We claim that M is finite. For this we observe that by the definition of the $c_{i_1, \ldots, i_r}(k)$ in Lemma 4.2, all integrals in the sum in the definition of M are equal. It is thus sufficient to consider the summand with coefficient $c_{1,2,\ldots,r}(k)$. On the other hand, we write $k \in K$ as

$$k = \begin{pmatrix} k_1 & * \\ * & * \end{pmatrix}$$

with $r \times r$-block k_1 and observe that

$$\int_K \ln(c_{1,2,\ldots,r}(k)^{-1})\, dk = -\int_K \ln \Delta_r \left(k^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} k \right) dk = -\int_K \ln \det(k_1^* k_1)\, dk,$$

which is finite as a consequence of Lemma 4.4. Therefore, M is finite which proves (2).

Finally, (3) is a consequence of (2). □

Lemma 5.2(3) implies that there exists a constant $C = C(q) > 0$ such that for all $t \in C_q^A$, $\lambda \in \mathbb{R}^q$,

$$|e^{i\langle \lambda, t \rangle} - e^{i\langle \lambda, m_{\mathbf{1}}(t) \rangle}| \le C \cdot \|\lambda\|. \tag{5.11}$$

Therefore, we conclude from Proposition 2.1(3):

5.3 Corollary. *There exists a constant $C = C(q) > 0$ such that for all $t \in C_q^A$, $\lambda \in \mathbb{R}^q$,*

$$\|\varphi_{-i\rho-\lambda}(t) - e^{i\langle\lambda,t\rangle}\| \le C \cdot (\|\lambda\| + \|\lambda\|^2).$$

We next turn to the proof of Proposition 2.2.

Proof of Proposition 2.2. Let $t \in C_q^A$. Consider a non-trivial row vector $a = (a_1, \ldots, a_q) \in \mathbb{R}^q \setminus \{0\}$ as well as the continuous functions

$$f_1(k) := \ln \Delta_1(k^* e^{2\underline{t}} k) \quad \text{and} \quad f_l(k) := \ln \Delta_l(k^* e^{2\underline{t}} k) - \ln \Delta_{l-1}(k^* e^{2\underline{t}} k) \quad (l = 2, \cdots, q).$$

Then, by (2.6), (2.7), (2.8), and the Cauchy-Schwarz inequality,

$$a\left(m_{\mathbf{2}}(t) - m_{\mathbf{1}}(t)^t m_{\mathbf{1}}(t)\right) a^t = \int_K \left(\sum_{l=1}^q a_l f_l(k)\right)^2 dk - \left(\int_K \sum_{l=1}^q a_l f_l(k)\, dk\right)^2 \ge 0. \tag{5.12}$$

This shows part (1) of the proposition. Moreover, for $t = c \cdot (1, \ldots, 1) \in C_q^A$ with $c \in \mathbb{R}$, the functions f_l are constant on K for all $l = 1, \ldots, q$ which implies $\Sigma^2(t) = 0$ and thus part (2).

For the proof of part (3) we notice that we have equality in (5.12) if and only if the function

$$k \mapsto \sum_{l=1}^q a_l f_l(k) = (a_1 - a_2) \ln \Delta_1(k^* e^{2\underline{t}} k) + \cdots + (a_{q-1} - a_q) \ln \Delta_{q-1}(k^* e^{2\underline{t}} k) + a_q \ln \Delta_q(k^* e^{2\underline{t}} k)$$

is constant on K. As $k \mapsto \ln \Delta_q(k^* e^{2\underline{t}} k)$ is constant on K, and as under the condition of (3), the functions $k \mapsto \ln \Delta_r(k^* e^{2\underline{t}} k)$ $(r = 1, \ldots, q-1)$ and the constant function 1 are linearly independent on K by Lemma 4.6, the function $k \mapsto \sum_{l=1}^q a_l f_l(k)$ is constant on K precisely for $a_1 = a_2 = \ldots = a_q$. This proves that $\Sigma^2(t)$ has rank $q-1$ as claimed.

We next turn to part (4). We recall that Lemma 4.2 implies

$$2jt_q \le \ln \Delta_j(k^* e^{2\underline{t}} k) \le 2jt_1$$

for $k \in K$, $t \in C_q^A$, and $j = 1, \ldots, q$. Therefore, by the integral representation (2.6),

$$\begin{aligned} |m_{j,l}(t)| \le& \frac{1}{4} \int_K \left|\ln \Delta_j(k^* e^{2\underline{t}} k) - \ln \Delta_{j-1}(k^* e^{2\underline{t}} k)\right| \cdot \left|\ln \Delta_l(k^* e^{2\underline{t}} k) - \ln \Delta_{l-1}(k^* e^{2\underline{t}} k)\right| dk \\ &\le \big((j-1)(t_1 - t_q) + \max(|t_1|, |t_q|)\big)\big((l-1)(t_1 - t_q) + \max(|t_1|, |t_q|)\big) \end{aligned}$$

for $j, l = 1, \ldots, q$ and $t \in C_q^A$. This implies part (4).

For the proof of part (5) we recall from the proof of Lemma 5.2(2) that for all $t \in C_q^A$ and $k \in K$,

$$0 \le 2t_1 - \ln \Delta_1(k^* e^{2\underline{t}} k) \le \ln M(k)$$

with $M(k) \le \infty$ as defined there for $r = 1$. This leads to

$$\begin{aligned} |(\ln \Delta_1(k^* e^{2\underline{t}} k))^2 - 4t_1^2| &= (2t_1 - \ln \Delta_1(k^* e^{2\underline{t}} k)) \cdot |\ln \Delta_1(k^* e^{2\underline{t}} k) + 2t_1| \\ &\le (2t_1 - \ln \Delta_1(k^* e^{2\underline{t}} k)) \cdot (4|t_1| + \ln M(k)) \\ &\le 4|t_1|(2t_1 - \ln \Delta_1(k^* e^{2\underline{t}} k)) + (\ln M(k))^2. \end{aligned}$$

Thus,

$$\left|\int_K (\ln \Delta_1(k^* e^{2\underline{t}} k))^2\, dk - 4t_1^2\right| \le 4|t_1|\Big(2t_1 - \int_K \ln \Delta_1(k^* e^{2\underline{t}} k)\, dk\Big) + \int_K (\ln M(k))^2\, dk.$$

As

$$2t_1 - \int_K \ln \Delta_1(k^* e^{2\underline{t}} k)\, dk$$

remains bounded for $t \in C_q^A$ by Lemma 5.2, and as $\int_K (\ln M(k))^2\, dk$ is finite as a consequence of Lemma 4.4 by the same arguments as in the end of the proof of Lemma 5.2, we see that for $t \in C_q^A$,

$$\Big|\int_K (\ln \Delta_1(k^* e^{2\underline{t}} k))^2\, dk - 4t_1^2\Big| \le C(|t_1|+1)$$

which proves the first inequality of Proposition 2.2 (5). For the proof of the second inequality, we again use the proof of Lemma 5.2(2) now for $r = q-1$. This and Lemma 5.2(1) lead to

$$0 \le \ln\left(\frac{\Delta_q(k^* e^{2\underline{t}} k)}{\Delta_{q-1}(k^* e^{2\underline{t}} k)}\right) - t_q \le M(k) \le \infty$$

for $k \in K$, $t \in C_q^A$. This implies the second inequality of Proposition 2.2 (5) in the same way as in the preceding case. □

We finally turn to the proof of Proposition 2.3 which is closely related to Proposition 2.2.

Proof of Proposition 2.3. Let $\nu \in M^1(C_q^A)$ with finite second moments. Consider a row vector $a = (a_1, \ldots, a_q) \in \mathbb{R}^q \setminus \{0\}$ as well as the continuous functions

$$f_1(k,t) := \ln \Delta_1(k^* e^{2\underline{t}} k) \quad \text{and} \quad f_l(k,t) := \ln \Delta_l(k^* e^{2\underline{t}} k) - \ln \Delta_{l-1}(k^* e^{2\underline{t}} k) \quad (l = 2, \cdots, q)$$

on $K \times C_q^A$. Then, by the definition of $\Sigma^2(\nu)$, (2.6), (2.7), (2.8), and the Cauchy-Schwarz inequality,

$$\begin{aligned} a\Sigma^2(\nu)a^t &= a\left(m_{\mathbf{2}}(\nu) - m_{\mathbf{1}}(\nu)^t m_{\mathbf{1}}(\nu)\right) a^t \\ &= \int_{C_q^A}\int_K \left(\sum_{l=1}^q a_l f_l(k,t)\right)^2 dk\, d\nu(t) - \left(\int_{C_q^A}\int_K \sum_{l=1}^q a_l f_l(k,t)\, dk\, d\nu(t)\right)^2 \ge 0 \qquad (5.13)\end{aligned}$$

where equality holds if and only if the continuous function

$$h : (k,t) \mapsto \sum_{l=1}^q a_l f_l(k,t) = (a_1 - a_2)\ln \Delta_1(k^* e^{2\underline{t}} k) + \cdots + (a_{q-1} - a_q)\ln \Delta_{q-1}(k^* e^{2\underline{t}} k) + a_q \ln \Delta_q(k^* e^{2\underline{t}} k)$$

is constant on $K \times C_q^A$ $\nu \otimes \omega_K$-almost surely with the uniform distribution ω_K on K. This just means that h is constant on $supp\,(\nu \otimes \omega_K) = (supp\,\nu) \times K$.

Assume now that ν satisfies the conditions of part (1) of the proposition, i.e., that $supp\,\nu \not\subset D_q := \{c\cdot(1, \ldots, 1) : c \in \mathbb{R}\} \subset C_q^A$, and that the orthogonal projection $\tau(\nu) \in M^1(D_q)$ of ν from C_q^A onto D_q is no point measure. Now choose $t \in supp\nu \setminus D_s$. As $h(t,.)$ is constant on K, we conclude from the proof of Proposition 2.2(3) that $a_1 = a_2 = \ldots = a_q$. Therefore, $h(k,t) = a_1 \cdot \ln \Delta_q(k^* e^{2\underline{t}} k) = a_1(t_1 + \ldots + t_q)$ is independent of t for $t \in supp\,\tau(\nu)$ which leads to $a_1 = 0$. This shows that under the conditions of part (3), $\Sigma^2(\nu)$ has full rank as claimed.

Parts (2) and (3) also follow by the same arguments and those of Proposition 2.2. □

6 Oscillatory behavior of hypergeometric functions of type BC at the identity

In this section we prove Propositions 3.2, 3.3, and 3.4 about the moment functions on the Weyl chamber C_q^B. The proofs are related to those for the A-case in Section 5.

We again start with the oscillatory behavior of hypergeometric functions φ_λ^p of type B at the identity for $p > 2q-1$. For this we recall and modify two results about principal minors and determinants from [RV1]. In our notation, Lemma 4.8 of [RV1] is as follows:

6.1 Lemma. *Let $t \in C_q^B$, $w \in B_q$, $u \in U(q,\mathbb{F})$ and $r = 1,\ldots,q$. Denote the ordered singular values of the $q\times q$-matrix w by $1 \ge \sigma_1(w) \ge \ldots \ge \sigma_q(w) \ge 0$. Then*

$$\frac{\Delta_r(g(t,u,w))}{\Delta_r(g(t,u,0))} \in \big[(1-\widetilde{t}\,\sigma_1(w))^{2r}, (1+\widetilde{t}\,\sigma_1(w))^{2r}\big], \quad \textit{with } \widetilde{t} := \min(t_1,1).$$

6.2 Lemma. *For each $p > 2q-1$ there exists $\epsilon > 0$ with*

$$\int_{B_q} \Delta(I - w^*w)^{-\epsilon} dm_p(w) < \infty.$$

Proof. The proof is similar to that of Lemma 4.10 in [RV1]. We consider the ball

$$B := \{y \in \mathbb{F}^q : \; \|y\|_2 < 1\}$$

and the diffeomorphism

$$P : B^q \longrightarrow B_q, \quad (y_1,\ldots,y_q) \longmapsto \begin{pmatrix} y_1 \\ y_2(I_q - y_1^*y_1)^{1/2} \\ \vdots \\ y_q(I_q - y_{q-1}^*y_{q-1})^{1/2}\cdots(I_q - y_1^*y_1)^{1/2}\end{pmatrix}; \tag{6.1}$$

see Lemma 3.7 and Corollary 3.8 of [R1] and Remark 2.6 of [RV1]. Using the transformation formula, these results also ensure that for some constant $\kappa > 0$,

$$\int_{B_q} \Delta(I - w^*w)^{-\epsilon} dm_p(w) = \frac{1}{\kappa}\int_{B^q} \prod_{j=1}^{q} (1 - \|y_j\|_2^2)^{d(p-q-j+1)/2-1-\epsilon} dy_1 \ldots dy_q.$$

The second integral is clearly finite for $d(p-2q+1)/2 - \epsilon > 0$, i.e., for $p - 2q + 1 > 2\epsilon/d$. This implies the claim. □

Proof of Proposition 3.2(3). Let $p > 2q-1$, $\lambda \in \mathbb{R}^q$, and $t \in C_q^B$. We use the integral representations (3.6) and (3.9) for the spherical functions and the associated moment functions $m_{\mathbf{1}}$ and study

$$\begin{aligned} R :=& R(\lambda,t) := |\varphi^p_{-i\rho-\lambda}(t) - e^{i\langle\lambda, m_{\mathbf{1}}(t)\rangle}| \qquad (6.2)\\ =&\Big|\int_{B_q}\int_{U(q,\mathbb{F})} exp\Big(\frac{i}{2}\sum_{r=1}^{q}(\lambda_r - \lambda_{r+1})\ln\Delta_r(g(t,u,w))\Big)\, du\, dm_p(w) \\ &\qquad - exp\Big(\frac{i}{2}\int_{B_q}\int_{U(q,\mathbb{F})}\sum_{r=1}^{q}(\lambda_r - \lambda_{r+1})\ln\Delta_r(g(t,u,w))\, du\, dm_p(w)\Big)\Big| \end{aligned}$$

with the convention $\lambda_{q+1} = 0$. We use the homogeneous polynomials C_r from (4.1) for $r = 1,\ldots,q$ and write the logarithms of the principal minors in (6.2) as

$$\ln\Delta_r(g(t,u,w)) = \ln C_r(\cosh^2 t_1,\ldots,\cosh^2 t_r) + \ln H_r(t,u,w) \tag{6.3}$$

with

$$H_r(t,u,w) := \frac{\Delta_r(g(t,u,w))}{C_r(\cosh^2 t_1,\ldots,\cosh^2 t_r)}. \tag{6.4}$$

Hence

$$\begin{aligned} R =&\Big|\int_{B_q}\int_{U(q,\mathbb{F})} exp\Big(\frac{i}{2}\sum_{r=1}^{q}(\lambda_r - \lambda_{r+1})\ln H_r(t,u,w)\Big)\, du\, dm_p(w) \\ &\qquad - exp\Big(\frac{i}{2}\int_{B_q}\int_{U(q,\mathbb{F})}\sum_{r=1}^{q}(\lambda_r - \lambda_{r+1})\ln H_r(t,u,w)\, du\, dm_p(w)\Big)\Big|. \end{aligned}$$

As before, the power series for both exponential functions lead to $R \le R_1 + R_2$ for

$$R_1 := \int_{B_q} \int_{U(q,\mathbb{F})} \Bigg| exp\left(\frac{i}{2}\sum_{r=1}^{q}(\lambda_r - \lambda_{r+1}) \ln H_r(t,u,w))\right) - \left(1 + \frac{i}{2}\sum_{r=1}^{q}(\lambda_r - \lambda_{r+1}) \cdot \ln H_r(t,u,w)\right)\Bigg| \, du \, dm_p(w),$$

$$R_2 := \Bigg| exp\left(\frac{i}{2}\int_{B_q} \int_{U(q,\mathbb{F})} \sum_{r=1}^{q}(\lambda_r - \lambda_{r+1}) \cdot \ln(H_r(t,u,w) \, dk\right) - 1 - \frac{i}{2}\int_{B_q} \int_{U(q,\mathbb{F})} \sum_{r=1}^{q}(\lambda_r - \lambda_{r+1}) \cdot \ln H_r(t,u,w)) \, du \, dm_p(w)\Bigg|.$$

We now use $|e^{ix} - (1+ix)| \le x^2$ for $x \in \mathbb{R}$ again and define

$$A_m := \int_{B_q} \int_{U(q,\mathbb{F})} \left|\sum_{r=1}^{q}(\lambda_r - \lambda_{r+1}) \cdot \ln H_r(t,u,w)\right|^m du \, dm_p(w) \qquad (m = 1,2).$$

Hence, by Jensen's inequality,

$$R \le R_1 + R_2 \le A_2 + A_1^2 \le 2A_2.$$

As

$$A_2 \le \|\lambda\|^2 \cdot const. \cdot \int_{B_q} \int_{U(q,\mathbb{F})} \sum_{r=1}^{q} |\ln H_r(t,u,w)|^2 \, dk =: \|\lambda\|^2 \cdot B_2,$$

we obtain $R \le B_2 \cdot 2\|\lambda\|^2$. To complete the proof, we check that B_2, i.e., the integrals

$$L_r := \int_{B_q} \int_{U(q,\mathbb{F})} |\ln H_r(t,u,w)|^2 \, du \, dm_p(w) \tag{6.5}$$

remain bounded independent of t for $r = 1, \ldots, q$. For this fix r and recall that by (6.4),

$$\ln H_r(t,u,w) = \ln \Delta_r(g(t,u,w)) - \ln C_r(\cosh^2 t_1, \ldots, \cosh^2 t_r).$$

Moreover, by Lemma 6.1

$$\ln \Delta_r(g(t,u,w)) - \ln \Delta_r(g(t,u,0)) \in 2r[\ln(1-\sigma_1(w)), \ln(1+\sigma_1(w))].$$

Thus,

$$|\ln H_r(t,u,w)|^2 \le 2\left|\ln\left(\frac{\Delta_r(g(t,u,0))}{C_r(\cosh^2 t_1, \ldots, \cosh^2 t_r)}\right)\right|^2 + 8r^2(|\ln(1+\sigma_1(w))|^2 + |\ln(1-\sigma_1(w))|^2). \tag{6.6}$$

Moreover, by the definition of B_q,

$$\int_{B_q} \int_{U(q,\mathbb{F})} |\ln(1+\sigma_1(w))|^2 \, du \, dm_p(w) \le (\ln 2)^2. \tag{6.7}$$

To handle the more critical term $|\ln(1-\sigma_1(w))|^2$, we use the elementary fact that for all $\epsilon > 0$ and $x \in]0,1[$, $|\ln x| \le x^{-\epsilon}$. As for $w \in B_q$, $1 \ge \sigma_1(w) \ge \ldots \ge \sigma_q(w) \ge 0$, we get

$$\frac{1}{1-\sigma_1(w)} \le \frac{2}{1-\sigma_1(w)^2} \le 2\prod_{r=1}^{q} \frac{1}{1-\sigma_r(w)^2} = \frac{2}{\Delta(I - w^*w)}. \tag{6.8}$$

We thus obtain that for all $\epsilon > 0$,

$$|\ln(1-\sigma_1(w))|^2 \leq (1-\sigma_1(w))^{-2\epsilon} \leq 2^{2\epsilon} \cdot \Delta(I - w^*w)^{-2\epsilon}.$$

Thus, by Lemma 6.2,

$$\int_{B_q} \int_{U(q,\mathbb{F})} |\ln(1-\sigma_1(w))|^2 \, du \, dm_p(w) \leq const. \cdot \int_{B_q} \Delta(I - w^*w)^{-2\epsilon} \, dm_p(w) < \infty. \tag{6.9}$$

It is therefore sufficient to prove that

$$\int_{B_q} \int_{U(q,\mathbb{F})} \left| \ln \left(\frac{\Delta_r(g(t,u,0))}{C_r(\cosh^2 t_1, \ldots, \cosh^2 t_r)} \right) \right|^2 du \, dm_p(w) \tag{6.10}$$

remains bounded independent of t. But this integral is equal to

$$\int_{U(q,\mathbb{F})} \left| \ln \left(\frac{\Delta_r(u^* \cosh^2 tu)}{C_r(\cosh^2 t_1, \ldots, \cosh^2 t_r)} \right) \right|^2 du, \tag{6.11}$$

and this expression remains bounded independent of t by the proof of Proposition 2.1(3) in Section 5; see Eqs. (5.2) and (5.4) and the arguments after (5.4) there. This completes the proof. □

For the case $q = 1$, Proposition 3.2(3) was proved in [V2] by the same approach in the context of Jacobi functions; see also [Z1], [Z2] for the context of Sturm-Liouville hypergroups.

We now turn to the proof of the remaining parts of Proposition 3.2. Part (1) follows from the LLN 3.5(1). Notice that the proof of this LLN in Section 8 is independent from Proposition 3.2(1). For the proof of part (2) we state the following result, which is related to estimates in the proof of Proposition 3.2(1), and which reduces estimates from the BC-case to the A-case in Section 5.

6.3 Lemma. *For $r = 1, \ldots, q$, $t \in C_q^B$, $u \in U(q,\mathbb{F})$, and $w \in B_q$,*

$$\left| \ln \Delta_r(g(t,u,w)) - \ln \Delta_r(u^* e^{2\underline{t}} u) \right| \leq \ln 4 + 2r \cdot \max\bigl(|\ln(1-\sigma_1(w))|, \ln(1+\sigma_1(w))\bigr)$$

with

$$\int_{B_q} \max\bigl(|\ln(1-\sigma_1(w))|, \ln(1+\sigma_1(w))\bigr) \, dm_p(w) < \infty.$$

Proof. We conclude from Lemma 6.1 that for $u \in U(q,\mathbb{F})$ and $w \in B_q$,

$$\Delta_r(g(t,u,0))(1-\sigma_1(w))^{2r} \leq \Delta_r(g(t,u,w)) \leq \Delta_r(g(t,u,0))(1+\sigma_1(w))^{2r}$$

and thus

$$\left| \ln \Delta_r(g(t,u,w)) - \ln \Delta_r(g(t,u,0)) \right| \leq 2r \cdot \max\bigl(|\ln(1-\sigma_1(w))|, \ln(1+\sigma_1(w))\bigr). \tag{6.12}$$

Moreover, as

$$\frac{1}{4} u^* e^{2\underline{t}} u \leq u^* (\cosh \underline{t})^2 u \leq u^* e^{2\underline{t}} u,$$

we have

$$\left| \ln \Delta_r(g(t,u,0)) - \ln \Delta_r(u^* e^{2\underline{t}} u) \right| \leq \ln 4$$

for $t \in C_q^B$, $u \in U(q,\mathbb{F})$. In combination with (6.12), this leads to the first estimation of the lemma. For the second statement, we first observe that $\int_{B_q} \ln(1+\sigma_1(w)) \, dm_p(w)$ is obviously finite. Moreover, $\int_{B_q} |\ln(1-\sigma_1(w))| \, dm_p(w)$ is also finite as a consequence of (6.9). □

Proposition 3.2(2) is now part (2) of the following result:

6.4 Lemma. (1) *For $r = 1, \ldots, q$ let*

$$s_r^{BC}(t) := m_{(1,0,\ldots,0)}(t) + \cdots + m_{(0,\ldots,0,1,0,\ldots,0)}(t) \qquad \text{for} \quad t \in C_q^B$$

be the sum of the first r moment functions of first order. Then there is a constant $C = C(q)$ such that for all $r = 1, \ldots, q$ and $t \in C_q^B$,

$$|t_1 + t_2 + \cdots + t_r \, - \, s_r^{BC}(t)| \leq C.$$

(2) *There is a constant $C = C(q)$ such that for all $t \in C_q^A$*

$$\|t - m_{\mathbf{1}}(t)\| \leq C.$$

Proof. Let $t \in C_q^B$. By the integral representation (3.9) of the moment functions, we have

$$s_r^{BC}(t) = \frac{1}{2} \int_{B_q} \int_{U(q,\mathbb{F})} \ln \Delta_r(g(t,u,w)) \, du \, dm_p(w) \qquad (r = 1, \ldots, q). \tag{6.13}$$

We thus obtain from Lemma 6.3 that for all $t \in C_q^B$ and $r = 1, \ldots, q$,

$$\left| s_r^{BC}(t) - \frac{1}{2} \int_{U(q,\mathbb{F})} \ln \Delta_r(u^* e^{2\underline{t}} u) \, du \right| \leq C$$

for some constant $C > 0$. Therefore, in the notation of Lemma 5.2,

$$\left| s_r^{BC}(t) - s_r(t) \right| \leq C \qquad (t \in C_q^B, \;\; r = 1, \ldots, q).$$

Lemma 5.2(2) now implies that for all $t \in C_q^B$ and $r = 1, \ldots, q$,

$$|s_r(t) - (t_1 + \ldots t_r)| \leq \tilde{C}$$

for some constant $\tilde{C}$. This proves part (1). Part (2) is a consequence of part (1). $\square$

6.5 Remark. We conjecture that in part (1) of the preceding lemma the stronger result

$$0 \leq t_1 + \ldots t_r - s_r^{BC}(t) \leq C \qquad (r = 1, \ldots, q, t \in C_q^B) \tag{6.14}$$

holds which would correspond to Lemma 5.2(2) in the A-case.

In fact, this could be easily derived from the attempting matrix inequality

$$(\cosh \underline{t} + \sinh \underline{t} \cdot w)(\cosh \underline{t} + \sinh \underline{t} \cdot w)^* \leq e^{2\underline{t}} \qquad (t \in C_q^B, w \in B_q).$$

Unfortunately, this matrix inequality is not correct. Take for instance $q = 2$, $t = (t_1, 0)$ with t_1 large, and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, a proof of (6.14) would be more involved than in the A-case above.

We next turn to the proof of Proposition 3.3.

Proof of Proposition 3.3. Fix $t \in C_q^B$. Consider a non-trivial row vector $a = (a_1, \ldots, a_q) \in \mathbb{R}^q \setminus \{0\}$ and the continuous functions

$$f_1(u,w) := \ln \Delta_1(g(t,u,w)) \quad \text{and} \quad f_l(u,w) := \ln \Delta_l(g(t,u,w)) - \ln \Delta_{l-1}(g(t,u,w)) \quad (l = 2, \cdots, q)$$

on $U(q,\mathbb{F}) \times B_q$. Then, by (3.9), (3.10), (3.11), and the Cauchy-Schwarz inequality,

$$\begin{aligned} & a \left(m_{\mathbf{2}}(t) - m_{\mathbf{1}}(t)^t m_{\mathbf{1}}(t) \right) a^t \qquad\qquad (6.15) \\ &= \int_{B_q} \int_{U(q,\mathbb{F})} \left(\sum_{l=1}^q a_l f_l(u,w) \right)^2 du \, dm_p(w) - \left(\int_{B_q} \int_{U(q,\mathbb{F})} \sum_{l=1}^q a_l f_l(u,w) \, du \, dm_p(w) \right)^2 \geq 0. \end{aligned}$$

This shows part (1) of the proposition.

For the proof of (2), use that for $t = 0 \in C_q^B$, $m_{\mathbf{1}}(0) = 0$ and $m_{\mathbf{2}}(0) = 0$ which yields $\Sigma^2(t) = 0$.

For the proof of part (3) we take $t \in C_q^B$ with $t \neq 0$ and notice that we have equality in (6.15) if and only if the function

$$\begin{aligned}(u,w) \mapsto &\sum_{l=1}^{q} a_l f_l(u,w) \\ &= (a_1 - a_2)\ln \Delta_1(g(t,u,w)) + \cdots + (a_{q-1} - a_q)\ln \Delta_{q-1}(g(t,u,w)) + a_q \ln \Delta_q(g(t,u,w))\end{aligned}$$

is constant on $U(q,\mathbb{F}) \times B_q$. Assume now that this is the case.

We now first consider the case where $t \in C_q^B$ does not have the form $t = c(1,\dots,1)$ with some $c > 0$. In this case we put $w = I_q \in B_q$ with $g(t,u,I_q) = u^* e^{2\underline{t}} u$. Therefore,

$$u \mapsto \ln \Delta_q(u^* e^{2\underline{t}} u)$$

is constant on $U(q,\mathbb{F})$, and by our assumption and Lemma 4.6, the functions $u \mapsto \ln \Delta_r(u^* e^{2\underline{t}} u)$ $(r = 1,\dots,q-1)$ and the constant function 1 are linearly independent on $U(q,\mathbb{F})$. Consequently, as the function

$$(u,w) \mapsto \sum_{l=1}^{q} a_l f_l(u,w)$$

is constant on $U(q,\mathbb{F}) \times B_q$, we have $a_1 = a_2 = \ldots = a_q$. On the other hand,

$$\ln \Delta_q(g(t,u,w)) = \ln |\Delta(\cosh \underline{t} + \sinh \underline{t} \cdot w)|^2$$

is not constant in $w \in B_q$ for $t \neq 0$, which proves $a_q = 0$. This shows that $\Sigma^2(t)$ is positive definite for $t \in B_q$ not having the form $c(1,\dots,1)$. Finally, if t has the form $t = c(1,\dots,1)$ with some $c > 0$, we may choose $w = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in B_q$ (with $1 \in \mathbb{R}$). Then $g(t,u,w) = u^* D(t) u$ with some diagonal matrix $D(t)$ where not all diagonal entries are equal. As above, Lemms 4.6 yields in this case that $a_1 = a_2 = \ldots = a_q$ and the proof can be completed in the same way as in the preceding case.

We next turn to part (4) of the proposition. We recall that Lemma 4.2 implies

$$2jt_q \le \ln \Delta_j(u^* e^{2\underline{t}} u) \le 2jt_1$$

for $u \in U(q,\mathbb{F})$, $t \in C_q^B$, and $j = 1,\dots,q$. Therefore, by the integral representation (3.9) of the moment functions and by Lemma 6.3

$$\begin{aligned}|m_{j,l}(t)| \le &\frac{1}{4}\int_{B_q}\int_{U(q,\mathbb{F})} \Big|\ln \Delta_j(g(t,u,w)) - \ln \Delta_{j-1}(g(t,u,w))\Big| \cdot \\ &\qquad \cdot \Big|\ln \Delta_l(g(t,u,w)) - \ln \Delta_{l-1}(g(t,u,w))\Big|\, du\, dm_p(w) \\ \le &C + \frac{1}{4}\int_{U(q,\mathbb{F})} \Big|\ln \Delta_j(u^* e^{2\underline{t}} u) - \ln \Delta_{j-1}(u^* e^{2\underline{t}} u)\Big| \cdot \Big|\ln \Delta_l(u^* e^{2\underline{t}} u) - \ln \Delta_{l-1}(u^* e^{2\underline{t}} u)\Big|\, du \\ \le &C(1 + t_1)\end{aligned}$$

for $j,l = 1,\dots,q$, $t \in C_q^B$, and some constant $C > 0$. On the other hand, by the definition of $g(t,u,w)$, the functions $\ln \Delta_l(g(t,u,w))$ are analytic at $t = 0$ with $\ln \Delta_l(g(0,u,w)) = 0$. Therefore, $m_{j,l}(t) = O(t_1^2)$ for small $t \in C_q^B$. We thus obtain that $|m_{j,l}(t)| \le t_1^2$ for all $t \in C_q^B$ and j,l with some constant $C > 0$ as claimed in part (4).

For the proof of part (5), we recall from the proof of Lemma 6.4(2) above that for all $t \in C_q^B$, $u \in U(q,\mathbb{F})$, and $w \in B_q$,

$$|2t_1 - \ln \Delta_1(g(t,u,w))| \le \ln M(u,w)$$

with some expression $M(u,w) \le \infty$ satisfying $\int_{U(q,\mathbb{F})} \int_{B_q} \ln M(u,w)\, du\, dm_p(w) < \infty$. This leads to

$$\begin{aligned} |(\ln \Delta_1(g(t,u,w)))^2 - 4t_1^2| &= (2t_1 - \ln \Delta_1(g(t,u,w))) \cdot |\ln \Delta_1(g(t,u,w)) + 2t_1| \\ &\le (2t_1 - \ln \Delta_1(g(t,u,w))) \cdot (4|t_1| + \ln M(u,w)) \\ &\le 4|t_1|(2t_1 - \ln \Delta_1(g(t,u,w))) + (\ln M(u,w))^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\Big|\int_{U(q,\mathbb{F})} \int_{B_q} (\ln \Delta_1(g(t,u,w)))^2\, du\, dm_p(w) - 4t_1^2\Big| \le \\ &\quad \le 4|t_1|\Big(2t_1 - \int_{U(q,\mathbb{F})} \int_{B_q} \ln \Delta_1(g(t,u,w))\, du\, dm_p(w)\Big) + \int_{U(q,\mathbb{F})} \int_{B_q} (\ln M(u,w))^2\, du\, dm_p(w). \end{aligned}$$

Therefore, as

$$\Big(2t_1 - \int_{U(q,\mathbb{F})} \int_{B_q} \ln \Delta_1(g(t,u,w))\, du\, dm_p(w)\Big)$$

remains bounded for $t \in C_q^B$ by Lemma 6.4, and as also

$$\int_{U(q,\mathbb{F})} \int_{B_q} (\ln M(u,w))^2\, du\, dm_p(w) < \infty$$

by the arguments of the proof of Lemma 5.2, we conclude that for $t \in C_q^B$,

$$\Big|\int_{U(q,\mathbb{F})} \int_{B_q} (\ln \Delta_1(g(t,u,w)))^2\, du\, dm_p(w) - 4t_1^2\Big| \le C(|t_1| + 1)$$

as claimed in Proposition 3.3(5). □

We finally turn to the proof of Proposition 3.4 which is closely related to Proposition 3.3(3).

Proof of Proposition 3.4. Let $\nu \in M^1(C_q^B)$ with finite second moments and $\nu \ne \delta_0$. Consider a row vector $a = (a_1, \ldots, a_q) \in \mathbb{R}^q \setminus \{0\}$ as well as the continuous functions

$$f_1(u,w,t) := \ln \Delta_1(g(t,u,w)) \quad \text{and} \quad f_l(k,t) := \ln \Delta_l(g(t,u,w)) - \ln \Delta_{l-1}(g(t,u,w)) \quad (l = 2, \cdots, q)$$

on $U(q,\mathbb{F}) \times B_q \times C_q^B$. By the definition of $\Sigma^2(\nu)$, (3.9), (3.10), (3.11), and the Cauchy-Schwarz inequality,

$$\begin{aligned} a\Sigma^2(\nu)a^t &= a\left(m_{\mathbf{2}}(\nu) - m_{\mathbf{1}}(\nu)^t m_{\mathbf{1}}(\nu)\right) a^t \\ &= \int_{C_q^B} \int_{U(q,\mathbb{F})} \int_{B_q} \left(\sum_{l=1}^q a_l f_l(k,t)\right)^2 dm_p(w)\, du\, d\nu(t) \\ &\qquad - \left(\int_{C_q^B} \int_{U(q,\mathbb{F})} \int_{B_q} \sum_{l=1}^q a_l f_l(k,t)\, dm_p(w)\, du\, d\nu(t)\right)^2 \quad \ge \quad 0 \end{aligned} \tag{6.16}$$

where equality holds if and only if the continuous function

$$h : (u,w,t) \mapsto \sum_{l=1}^q a_l f_l(u,w,t)$$

is constant on $K \times C_q^A$ almost surely w.r.t. $\omega_{U(q,\mathbb{F})} \times m_p \times \nu$-almost surely. This however, is not the case by the proof of Proposition 3.3(3). □

7 Proof of the stochastic limit theorems in the case A

In this section we prove the strong law of large numbers 2.4 and the CLT 2.5 for $K := U(q,\mathbb{F})$-biinvariant random walks $(S_k)_{k\geq 0}$ on $G := GL(q,\mathbb{F})$ associated with some $\nu \in M^1(C_q^A)$:

We first turn to the CLT 2.5. Besides the results of Section 5 we need the following estimate which follows immediately from the integral representation (2.5) for the functions φ_λ^A.

7.1 Lemma. *For all $t \in C_q^A$, $\lambda \in \mathbb{R}^q$, and $l \in \mathbb{N}_0^q$,*

$$\left|\frac{\partial^{|l|}}{\partial \lambda^l}\varphi^A_{-i\rho-\lambda}(t)\right| \leq m_l(t).$$

Let $m \in \mathbb{N}_0$. We say that $\nu \in M^1(C_q^A)$ admits finite m-th modified moments if in the notation of Section 2,

$$m_{(m,0,\ldots,0)}, m_{(0,m,0,\ldots,0)}, \ldots, m_{(0,\ldots,0,m)} \in L^1(C_q^A, \nu).$$

It follows from the integral representation (2.6) of the moment function and Hölder's inequality that in this case all moment functions of order at most m are ν-integrable. Moreover, this moment condition implies a corresponding differentiability of the spherical Fourier transform of ν:

7.2 Lemma. *Let $m \in \mathbb{N}_0$ and $\nu \in M^1(C_q^A)$ with finite m-th moments. Then the spherical Fourier transform*

$$\tilde{\nu} : \mathbb{R}^q \to \mathbb{C}, \quad \lambda \mapsto \int_{C_q^A} \varphi^A_{-i\rho-\lambda}(t)\, d\nu(t)$$

is m-times continuously partially differentiable, and for all $l \in \mathbb{N}_0^n$ with $|l| \leq m$,

$$\frac{\partial^{|l|}}{\partial \lambda^l}\tilde{\nu}(\lambda) = \int_{C_q^A} \frac{\partial^{|l|}}{\partial \lambda^l}\varphi^A_{-i\rho-\lambda}(t)\, d\nu(t). \tag{7.1}$$

In particular,

$$\frac{\partial^{|l|}}{\partial \lambda^l}\tilde{\nu}(0) = (-i)^{|l|}\int_{C_q^A} m_l(t)\, d\nu(t). \tag{7.2}$$

Proof. We proceed by induction: The case $m = 0$ is trivial, and for $m \to m+1$ we observe that by our assumption all moments of lower order exist, i.e., (7.1) is available for all $|l| \leq m$. It follows from Lemma 7.1 and a well-known result about parameter integrals that a further partial derivative and integration can be interchanged. Finally, (7.2) follows from (7.1) and (2.6). Continuity of the derivatives is also clear by Lemma 7.1. □

We now turn to the proof of the CLT:

Proof of Theorem 2.5. Let $\nu \in M^1(C_q^A)$ be a probability measure with finite second modified moments. Let $(X_k)_{k\geq 1}$ be i.i.d. G-valued random variables with the associated K-biinvariant distribution $\nu_G \in M^1(G)$ and $S_k := X_1 \cdot X_2 \cdots X_k$ as in Section 2. We consider the canonical projection

$$(\tilde{S}_k := \ln \sigma_{sing}(S_k))_{k\geq 0}$$

of this random walk from G to $G//K \simeq C_q$ as in Section 2.

Let $\lambda \in \mathbb{R}^q$. As the functions $\varphi^A_{-i\rho-\lambda}$ are bounded on C_q^A (by the integral representation (2.5)) and multiplicative w.r.t. double coset convolutions of measures on C_q^A, we have

$$E(\varphi^A_{-i\rho-\lambda/\sqrt{k}}(\tilde{S}_k)) = \int_{C_q^A} \varphi^A_{-i\rho-\lambda/\sqrt{k}}(t)\, d\nu^{(k)}(t) = \left(\int_{C_q^A} \varphi^A_{-i\rho-\lambda/\sqrt{k}}(t)\, d\nu(t)\right)^k = \tilde{\nu}(\lambda/\sqrt{k})^k.$$

We now use Taylor's formula, Lemma 7.2, and

$$m_{\mathbf{2}}(\nu) := \int_G m_{\mathbf{2}}(g)\, d\nu(g) = \Sigma^2(\nu) + m_{\mathbf{1}}(\nu)^t m_{\mathbf{1}}(\nu)$$

and obtain

$$\begin{aligned}
exp(i\langle\lambda, m_{\mathbf{1}}(\nu)\rangle\cdot\sqrt{k})\cdot E(\varphi^A_{-i\rho-\lambda/\sqrt{k}}(\tilde{S}_k)) &= \Big(exp(i\langle\lambda, m_{\mathbf{1}}(\nu)\rangle/\sqrt{k})\cdot\tilde{\nu}(\lambda/\sqrt{k})\Big)^k \qquad (7.3)\\
=\Bigg(\Big[1+\frac{i\langle\lambda, m_{\mathbf{1}}(\nu)\rangle}{\sqrt{k}} - \frac{\langle\lambda, m_{\mathbf{1}}(\nu)\rangle^2}{2k} + o(\frac{1}{k})\Big]\cdot\Big[1-\frac{i\langle\lambda, m_{\mathbf{1}}(\nu)\rangle}{\sqrt{k}} - \frac{\lambda m_{\mathbf{2}}(\nu)\lambda^t}{2k} + o(\frac{1}{k})\Big]\Bigg)^k\\
=\Bigg(\Big[1+\frac{i\langle\lambda, m_{\mathbf{1}}(\nu)\rangle}{\sqrt{k}} - \frac{(\langle\lambda, m_{\mathbf{1}}(\nu)\rangle^2}{2k} + o(\frac{1}{k})\Big]\cdot\\
\times\Big[1-\frac{i\langle\lambda, m_{\mathbf{1}}(\nu)\rangle}{\sqrt{k}} - \frac{\lambda(\Sigma^2(\nu)+m_{\mathbf{1}}(\nu)^t m_{\mathbf{1}}(\nu))\lambda^t}{2k} + o(\frac{1}{k})\Big]\Bigg)^k\\
=\Big(1-\frac{\lambda\Sigma^2(\nu)\lambda^t}{2k} + o(\frac{1}{k})\Big)^k.
\end{aligned}$$

Therefore,

$$\lim_{k\to\infty} exp(i\langle\lambda, m_{\mathbf{1}}(\nu)\rangle\sqrt{k})\cdot E(\varphi^A_{-i\rho-\lambda/\sqrt{k}}(\tilde{S}_k)) = exp(-\lambda\Sigma^2(\nu)\lambda^t/2).$$

Moreover, by Proposition 2.1(3)

$$\lim_{k\to\infty} E\left(\varphi^A_{-i\rho-\lambda/\sqrt{k}}(\tilde{S}_k) - exp(-i\langle\lambda, m_{\mathbf{1}}(\tilde{S}_k)\rangle/\sqrt{k})\right) = 0.$$

Therefore,

$$\lim_{k\to\infty} exp(-i\langle\lambda, (m_{\mathbf{1}}(\tilde{S}_k) - k\cdot m_{\mathbf{1}}(\nu))\rangle/\sqrt{k}) = exp(-\lambda\Sigma^2(\nu)\lambda^t/2)$$

for all $\lambda\in\mathbb{R}^q$. Levy's continuity theorem for the classical q-dimensional Fourier transform now implies that $(m_{\mathbf{1}}(\tilde{S}_k) - k\cdot m_{\mathbf{1}}(\nu))/\sqrt{k}$ tends in distribution to $N(0,\Sigma^2(\nu))$. By the estimate of Lemma 5.2(3), this shows that $(\tilde{S}_k - k\cdot m_{\mathbf{1}}(\nu))/\sqrt{k}$ tends in distribution to $N(0,\Sigma^2(\nu))$ as claimed. □

The oscillatory behavior of the φ^A_λ in Proposition 2.1(3) can be used to derive a Berry-Esseen-type estimate with the order $O(k^{-1/3})$ of convergence. As the details are technical and similar to the corresponding rank-one-case in Theorem 4.2 of [V3], we omit it. We also mention that Proposition 2.1(3) can be used to derive further CLTs e.g. with stable distributions as limits or a Lindeberg-Feller CLT. The details of proof then would be also very similar to the classical sums of iid random variables.

We next turn to the strong law of large numbers 2.4.

Proof of Theorem 2.4. We first prove part (2) and consider some $\nu\in M^1(C_q^A)$ having second moments. Let $\epsilon > 1/2$. We employ the strong law of large numbers 7.3.21 in [BH] for the random walk $(\tilde{S}_k)_{k\ge 0}$ on the double coset hypergroup $G//K \simeq C_q^A$ with the constants $r_k := k^{-2\epsilon}$ there which satisfy $\sum_{k=1}^\infty r_k < \infty$. For all $l = 1,\ldots,q$, we now apply this result to one-dimensional moment functions $m_{(0,\ldots,0,1,0,\ldots,0)}$ and $m_{(0,\ldots,0,2,0,\ldots,0)}$ of first and second order with the nontrivial entry in the position l. The integral representation (2.6) and Jensen's inequality ensure that

$$m^2_{(0,\ldots,0,1,0,\ldots,0)} \le m_{(0,\ldots,0,2,0,\ldots,0)} \qquad \text{on} \quad C_q^A,$$

i.e., condition (MF2) for Theorem 7.3.21 in [BH] holds. We conclude from this theorem that for all $l = 1,\ldots,q$ and vectors of the form $(0,\ldots,0,1,0,\ldots,0)$ with 1 at position l,

$$k^{-\epsilon}\cdot\big(m_{(0,\ldots,0,1,0,\ldots,0)}(\tilde{S}_k) - k\int_{C_q^A} m_{(0,\ldots,0,1,0,\ldots,0)}(t)\, d\nu(t)\big)$$

tends to 0 a.s. for $k\to\infty$. In other words, $k^{-\epsilon}\cdot\big(m_{\mathbf{1}}(\tilde{S}_k) - k\cdot m_{\mathbf{1}}(\nu)\big)$ tends to 0 a.s.. Proposition 2.1(2) finally implies that $k^{-\epsilon}\cdot\big(\tilde{S}_k - k\cdot m_{\mathbf{1}}(\nu)\big)$ tends to 0 a.s. as claimed.

Part (1) follows in the same way from Theorem 7.3.24 in [BH] with the constant $\lambda = 1$ there. □

8 Proof of the stochastic limit theorems in the case BC

In this section we prove the LLN 3.5 and the CLT 3.6 in the BC-case. Based on the technical results of Section 6, the proofs are very similar to those in Section 7. We therefore skip details.

We first turn to the CLT 3.6. Besides Section 6 we need the following immediate consequence of the integral representation (3.6) for φ_λ^p.

8.1 Lemma. *For all $t \in C_q^B$, $\lambda \in \mathbb{R}^q$, and $l \in \mathbb{N}_0^q$,*

$$\left| \frac{\partial^{|l|}}{\partial \lambda^l} \varphi^p_{-i\rho-\lambda}(t) \right| \le m_l(t).$$

Let $m \in \mathbb{N}_0$. We say that $\nu \in M^1(C_q^A)$ admits finite m-th modified moments if in the notation of Section 3,

$$m_{(m,0,\ldots,0)}, m_{(0,m,0,\ldots,0)}, \ldots, m_{(0,\ldots,0,m)} \in L^1(C_q^B, \nu).$$

By the integral representation (3.9) and by Hölder's inequality, in this case all moment functions of order at most m are ν-integrable. Moreover, this moment condition leads to a corresponding differentiability of the spherical Fourier transform of ν as in Lemma 7.2. We omit the proof:

8.2 Lemma. *Let $m \in \mathbb{N}_0$ and $\nu \in M^1(C_q^B)$ with finite m-th moments. Then the spherical Fourier transform*

$$\tilde{\nu} : \mathbb{R}^q \to \mathbb{C}, \quad \lambda \mapsto \int_{C_q^B} \varphi^p_{-i\rho-\lambda}(t)\, d\nu(t)$$

is m-times continuously partially differentiable, and for $l \in \mathbb{N}_0^n$ with $|l| \le m$,

$$\frac{\partial^{|l|}}{\partial \lambda^l} \tilde{\nu}(\lambda) = \int_{C_q^B} \frac{\partial^{|l|}}{\partial \lambda^l} \varphi^p_{-i\rho-\lambda}(t)\, d\nu(t). \tag{8.1}$$

In particular,

$$\frac{\partial^{|l|}}{\partial \lambda^l} \tilde{\nu}(0) = (-i)^{|l|} \int_{C_q^B} m_l(t)\, d\nu(t). \tag{8.2}$$

We now turn to the proof of the CLT:

Proof of Theorem 3.6. Let $\nu \in M^1(C_q^B)$ be a probability measure with finite second modified moments, $p \in [2q-1, \infty[$, and $d = 1, 2, 4$. As described in Section 3 we consider the associated time-homogeneous random walk $(\tilde{S}_k)_{k\ge 0}$ on C_q^B. Then, as described there, the distributions of $\tilde{S}_k$ are given as the convolution powers $\nu^{(k)}$ w.r.t. $*_p$. With this observation in mind, we can just use the results of Section 6 and Lemmas 8.1 and 8.2 instead of the results of Section 5 and Lemmas 7.1 and 7.2, respectively in order to complete the proof in the same way as for the CLT 2.5. $\square$

Finally, the strong law of large numbers 3.5 can be proved by the same methods as the strong law 2.4 in Section 7 by using the integral representation (3.9) of the moment functions instead of (2.6).

References

[BH] W.R. Bloom, H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups. *De Gruyter Studies in Mathematics* 20, de Gruyter-Verlag Berlin, New York 1995.

[BL] P. Bougerol, J. Lacroix, Products of Random Matrices with Applications to Schrödinger Operators, *Birkhäuser*, Boston, 1985.

[FH] J. Faraut, K. Harzallah, Distances hilbertiennes invariantes sur un espace homogene. *Ann. Inst. Fourier* 24, 171-217 (1974).

[F] J. Faraut, Dispersion d'une mesure de probabilité sur $SL(2,\mathbb{R})$ biinvariante par $SO(2,\mathbb{R})$ et théorème de la limite centrale. Universite Louis Pasteur, Strasbourg, 1975.

[G1] P. Graczyk, A central limit theorem on the space of positive definite symmetric matrices. *Ann. Inst. Fourier* 42, 857–874 (1992)

[G2] P. Graczyk, Dispersions and a central limit theorem on symmetric spaces. *Bull. Sci. Math.*, II. Ser., 118, 105-116 (1994).

[H] G. Heckman, Dunkl Operators. Séminaire Bourbaki 828, 1996–97; Astérisque 245 (1997), 223–246.

[HS] G. Heckman, H. Schlichtkrull, Harmonic Analysis and Special Functions on Symmetric Spaces. Perspect. Math. 16, Academic Press 1994.

[H1] S. Helgason, Groups and Geometric Analysis. Mathematical Surveys and Monographs, vol. 83, AMS 2000.

[H2] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces. AMS 2001.

[Ho] B. Hoogenboom, Spherical functions and invariant differential operators on complex Grassmann manifolds. *Ark. Mat.* 20 (1982), 69–85.

[HJ] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis. Cambridge University Press 1991.

[J] R.I. Jewett, Spaces with an abstract convolution of measures, *Adv. Math.* 18 (1975), 1–101.

[K] T. Koornwinder, Jacobi functions and analysis on noncompact semisimple Lie groups. In: *Special Functions: Group Theoretical Aspects and Applications*, Eds. Richard Askey et al., D. Reidel, Dordrecht-Boston-Lancaster, 1984.

[L] E. Le Page, Theoremes limites pour les produits de matrices aleatoires. In: Probability Measures on Groups, ed.: H. Heyer. Lecture Notes in Math. 928, Springer 1982, pp. 258–303.

[NPP] E. K. Narayanan, A. Pasquale, S. Pusti: Asymptotics of Harish-Chandra expansions, bounded hypergeometric functions associated with root systems, and applications. *Adv. Math.* 252 (2014), 227–259.

[O1] E.M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math.* 175, 75–112 (1995).

[O2] E.M. Opdam, Lecture Notes on Dunkl Operators for Real and Complex Reflection Groups. MSJ Memoirs, vol. 8, Mathematical Society of Japan, Tokyo, 2000.

[Ri] D.St.P. Richards, The central limit theorem on spaces of positive definite matrices. *J. Multiv. Anal.* 29, 326-332 (1989).

[R1] M. Rösler, Bessel convolutions on matrix cones, *Compos. Math.* 143 (2007), 749–779.

[R2] M. Rösler, Positive convolution structure for a class of Heckman-Opdam hypergeometric functions of type BC. *J. Funct. Anal.* 258 (2010), 2779–2800.

[RKV] M. Rösler, T. Koornwinder, M. Voit, Limit transition between hypergeometric functions of type BC and type A. *Compos. Math.* 149 (2013), 1381–1400.

[RV1] M. Rösler, M. Voit, Integral representation and uniform limits for some Heckman-Opdam hypergeometric functions of type BC. *Trans. Amer. Math. Soc.* 2015, DOI: http://dx.doi.org/10.1090/tran6673, to appear in print.

[RV2] M. Rösler, M. Voit, A central limit theorem for random walks on the dual of a compact Grassmannian. *Symmetry, Integrability and Geometry* SIGMA 11 (2015), 013, 18 pages, arXiv:1409.4213v3.

[Sa] P. Sawyer, Spherical functions on $SO_0(p,q)/SO(p) \times SO(q)$. *Canad. Math. Bull.* 42 (1999), 486–498.

[Sch] B. Schapira, Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel, *Geom. Funct. Anal.* 18 (2008), 222–250.

[Te1] A. Terras, Asymptotics of spherical functions and the central limit theorem on the space P_n of positive $n \times n$ matrices. *J. Multiv. Anal.* 23, 13-36 (1987).

[Te2] A. Terras, Harmonic Analysis on Symmetric Spaces and Applications II. Springer-Verlag 1988.

[Tu] V.N. Tutubalin, On the limit behaviour of compositions of measures in the plane and space of Lobachevski. *Theory Probab. Appl.* 7, 189–196 (1962).

[Vi] A.D. Virtser, Central limit theorem for semisimple Lie groups. *Theor. Probab. Appl.* 15, 667-687 (1970).

[V1] M. Voit, Central limit theorems for random walks on $\mathbb{N}_0$ that are associated with orthogonal polynomials. *J. Multivariate Anal.* 34, 290-322 (1990).

[V2] M. Voit, Bessel convolutions on matrix cones: Algebraic properties and random walks. *J. Theor. Probab.* 22 (2009), 741-771.

[V3] M. Voit, Central limit theorems for hyperbolic spaces and Jacobi processes on $[0, \infty[$. *Monatsh. Math.* 169, 441-468 (2013).

[V4] M. Voit, Product formulas for a two-parameter family of Heckman-Opdam hypergeometric functions of type BC. *J. Lie Theory* 25, 9-36 (2015).

[Z1] H. Zeuner, The central limit theorem for Chebli-Trimeche hypergroups. *J. Theor. Probab.* 2, 51-63 (1989).

[Z2] H. Zeuner, Moment functions and laws of large numbers on hypergroups. *Math. Z.* 211, 369–407 (1992).